DCF: Personal Income Tax
Valuation equation

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To value a levered firm we need to value an unlevered firm.

This remains true in the case of firm income tax as well as in the case of personal income tax.

Definition of cost of capital of unlevered firm as in Chapter 2, i.e.

$$\widetilde{k}_t^{E,\mathrm{u}} := \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}^{\mathrm{u}} + \widetilde{V}_{t+1}^{\mathrm{u}}\right]}{\widetilde{V}_t^{\mathrm{u}}} - 1 \; .$$

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Then, with deterministic cost of capital (without proof)

$$\widetilde{V}_t^{\mathrm{u}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{(1+k^{E,\mathrm{u}})^{s-t}} \ .$$

Again we assume

$$\mathrm{E}_t\left[\widetilde{CF}_{t+1}^{\mathrm{u}}\right] = (1+g)\widetilde{CF}_t^{\mathrm{u}} \ .$$

Personal income tax

The characteristics of personal income tax are:

- Tax subject: shareholder (and creditor).
- Tax base (categories of income):
  - shareholder: **dividends**, business income, . . .
    (capital repayment exempt from taxes)
  - debt holder: **interest**.
- Tax rate can be different for dividends (τ^D) and interest (τ^I).

From gross cash flow to tax base

Pre-tax gross cash flow	$\widetilde{GCF}_t$
$-$ Investment expenses	$\widetilde{Inv}_t$
$=$ Shareholder's unlevered taxable income	$\widetilde{GCF}_t - \widetilde{Inv}_t$
$-$ Retained earnings	$\widetilde{A}_t$
$+$ Cashflow from retained earnings	$(1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}$
$=$ Shareholder's levered taxable income	$\widetilde{GCF}_t - \widetilde{Inv}_t$ $-\widetilde{A}_t + (1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}$

How is the retained amount invested?

Machines, tools, buildings,...?

No! All projects with positive NPV are already realized.

Debt repayment?

No! Debt schedule is already given.

Capital market?

Yes! But at what interest rate??

- risk-free

$$\widetilde{r}_t = r_f$$

- risky

$$\widetilde{A}_t = \frac{\mathrm{E}_t^Q\left[(1+\widetilde{r}_t)\widetilde{A}_t\right]}{1+r_f} \Longrightarrow \mathrm{E}_t^Q\left[\widetilde{r}_t\right] = r_f$$

Tax due

- for the unlevered company

$$\widetilde{Tax}_t^{\mathrm{u}} = \tau^D(\widetilde{GCF}_t - \widetilde{Inv}_t)$$

- for the levered company

$$\widetilde{Tax}_t^{\mathrm{l}} = \tau^D\left(\widetilde{GCF}_t - \widetilde{Inv}_t - \widetilde{A}_t + (1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}\right).$$

Tax due of a levered company can be written as

$$\widetilde{Tax}_t^{\mathrm{l}} - \widetilde{Tax}_t^{\mathrm{u}} = \tau^D\Big(-\widetilde{A}_t + (1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}\Big).$$

Tax difference of levered firm

Tax due of the levered firm differs. Although unlevered and levered firms have identical

- gross cash flows,
- stock depreciation and appreciation,
- debt schedule and
- investments

they differ in

- dividends and
- tax payments.

If dividends are not fully distributed, the levered firm pays less personal income taxes than the unlevered firm. There is a

(personal income) tax shield for the levered firm – justifying our designation. The tax differences are

$$\begin{aligned}\widetilde{CF}^{\mathrm{l}}_t - \widetilde{CF}^{\mathrm{u}}_t &= \left(\ldots - \widetilde{A}_t + (1+\widetilde{r}_{t-1})\widetilde{A}_{t-1} - \widetilde{Tax}^{\mathrm{l}}_t\right) - \left(\ldots - \widetilde{Tax}^{\mathrm{u}}_t\right) \\ &= (1-\tau^D)\left(-\widetilde{A}_t + (1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}\right)\end{aligned}$$

and after taking expectations

$$\mathrm{E}^Q_{t-1}\left[\widetilde{CF}^{\mathrm{l}}_t - \widetilde{CF}^{\mathrm{u}}_t\right] = (1-\tau^D)\left((1+r_f)\widetilde{A}_{t-1} - \mathrm{E}^Q_{t-1}\left[\widetilde{A}_t\right]\right).$$

What is the fundamental theorem without any tax?

If capital markets are free of arbitrage, risk–neutral probabilities Q exist such that

$$V_t = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + r_f}.$$

What changed with **firm income tax**?

$\Longrightarrow$ Only payments $\widetilde{CF}_{t+1}$ were affected.

Fundamental theorem with personal income taxes11

What changes will there be with **personal income tax**?

$\Longrightarrow$ Both payments $\widetilde{CF}_{t+1}$ and interest r_f will be affected.

But how?! We do not prove here that

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + r_f(1-\tau^I)}$$

holds.

Gordon-Shapiro with personal income tax,

$$\widetilde{V}_t^{\mathrm{u}} = \frac{1+g}{k^{E,\mathrm{u}} - g} \widetilde{CF}_t^{\mathrm{u}}.$$

Cost of equity of unlevered firm is suitable as discount rates,

$$\frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 + r_f(1-\tau^I)\right)^{s-t}} = \frac{\mathrm{E}_t\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{(1+k^{E,\mathrm{u}})^{s-t}}.$$

Remark: Cost of equity $k^{E,\mathrm{u}}$ **after income tax**!

We know that the levered firm pays less taxes than the unlevered. But what is the value of this tax shield? And **what does this value depend upon?**

To this end we will assume that the last retainment at $T-1$ is zero, i.e. $\widetilde{A}_T = 0$.

The market value of the unlevered firm is

$$\widetilde{V}_t^{\mathrm{u}} = \frac{\mathsf{E}_t^Q\left[\widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1 + r_f(1-\tau^I)} + \ldots + \frac{\mathsf{E}_t^Q\left[\widetilde{CF}_{T}^{\mathrm{u}}\right]}{(1 + r_f(1-\tau^I))^{T-t}}.$$

The market value of the levered firm is

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathsf{E}_t^Q\left[\widetilde{CF}_{t+1}^{\mathrm{l}}\right]}{1 + r_f(1-\tau^I)} + \ldots + \frac{\mathsf{E}_t^Q\left[\widetilde{CF}_{T}^{\mathrm{l}}\right]}{(1 + r_f(1-\tau^I))^{T-t}}.$$

From both equations

$$\widetilde{V}_t^{\mathrm{l}} = \widetilde{V}_t^{\mathrm{u}} + \frac{(1-\tau^D)\,\mathrm{E}_t^Q\left[(1+r_f)\widetilde{A}_t - \widetilde{A}_{t+1}\right]}{1+r_f(1-\tau^I)} + \frac{(1-\tau^D)\,\mathrm{E}_t^Q\left[(1+r_f)\widetilde{A}_{t+1} - \widetilde{A}_{t+2}\right]}{1+r_f(1-\tau^I)^{T-t-1}} + \ldots + \frac{(1-\tau^D)\,\mathrm{E}_t^Q\left[(1+r_f)\widetilde{A}_{T-1} \overbrace{-\widetilde{A}_T}^{=0}\right]}{(1+r_f(1-\tau^I))^{T-t}}.$$

Rearranging,

$$\widetilde{V}_t^{\mathrm{l}} = \widetilde{V}_t^{\mathrm{u}} + \frac{(1-\tau^D)(1+r_f)\widetilde{A}_t}{1+r_f(1-\tau^I)}$$
$$+ \frac{\mathrm{E}_t^Q\left[\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_{t+1} - (1-\tau^D)\widetilde{A}_{t+1}\right]}{1+r_f(1-\tau^I)} + \ldots$$
$$+ \frac{\mathrm{E}_t^Q\left[\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_{T-1} - (1-\tau^D)\widetilde{A}_{T-1}\right]}{(1+r_f(1-\tau^I))^{T-1-t}}.$$

Which gives

$$\widetilde{V}_t^{\mathrm{l}} = \widetilde{V}_t^{\mathrm{u}} + \left(1-\tau^D\right)\widetilde{A}_t + \frac{\tau^I\left(1-\tau^D\right) r_f \, \mathrm{E}_t^Q\left[\widetilde{A}_t\right]}{1+r_f\left(1-\tau^I\right)} + \frac{\tau^I\left(1-\tau^D\right) r_f \, \mathrm{E}_t^Q\left[\widetilde{A}_{t+1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^2} + \ldots + \frac{\tau^I\left(1-\tau^D\right) r_f \, \mathrm{E}_t^Q\left[\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}}.$$

Compare this to the firm income tax:

$$\widetilde{V}_t^{\mathrm{l}} = \widetilde{V}_t^{\mathrm{u}} + \frac{\tau r_f \, \mathrm{E}_t^Q \left[\widetilde{D}_t\right]}{1 + r_f} + \ldots + \frac{\tau r_f \, \mathrm{E}^Q \left[\widetilde{D}_{T-1}\right]}{(1 + r_f)^{T-t}}.$$

Although a different economic story, a similar formal structure!

Only $\tau^I (1 - \tau^D) r_f \widetilde{A}_t$ replaces $\tau r_f \widetilde{D}_t$.

General procedure

We proceed as in Chapter 2:

1. We formulate different distribution policies.
2. We modify the main valuation equation with this distribution.

Different retainment policies

With **firm income tax** the **debt schedule** was important, with **personal income tax** the **retainment schedule** is important. We will look at

1. retainment based on cash flow,
2. retainment based on dividends,
3. retainment based on market values.

The fundamental theorem holds with an post-tax risk-free rate.

The distribution policy determines the value of the tax shield.