# DCF: Firm income tax
# Miles-Ezzell
# adjustment formula

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# Miles-Ezzell: the problem



Up to now we know three procedures to evaluate $\widetilde{E}_t$ (or $\widetilde{V}^{\mathrm{l}}_t$). In case of financing based on market values these procedures coincide, otherwise not.

But what can we tell about the relation between *WACC* and the unlevered cost of capital, i.e. $k^{E,\mathrm{u}}$?

This is the topic of **adjustment formulas**.

And this time we will need the assumption of martingale-like cash flows.

**Theorem (Miles-Ezzell 1980):** *If*

- *cash flows of the unlevered firm are martingale-like,*
- *the levered firm is financed based on market values and*
- *WACC is deterministic,*

*then*

$$1 + WACC_t = \left(1 + k_t^{E,\mathrm{u}}\right)\left(1 - \frac{\tau\, r_f}{1 + r_f}\,\mathrm{l}_t\right)$$

*and $k^{E,u}$ is deterministic.*

The original article of Miles-Ezzell required a constant leverage ratio $l_t$ and risk-free debt: **we do not!**

This adjustment formula finally shows why *WACC* might be useful (remember apples and oranges?).

The assumption of martingale-like cash flows is necessary (we come back to this).

The proof is not an easy task.

What is the connection to the Modigliani-Miller (1963) adjustment formula? Next lecture...

1. write down recursion formula with explicit tax shield (in nominator)
2. transfer tax shield to denominator (=first part of the proof)
3. *detour*:[1] rewrite recursion formula as infinite sum
4. apply fact that $\mathrm{E}^Q$ can be replaced by E if $r_f$ is replaced by cost of capital ("cost of capital are discount rates")
5. rewrite infinite sum as recursion equation (*detour finished*)
6. use definition of WACC

[1] The detour is necessary because our Theorem on discount rates can only be applied to cash flows $\widetilde{CF}^{\mathrm{u}}$. But in the recursion formula also $\widetilde{V}^{\mathrm{l}}$ appears!

# Proof (part I) 

We start from fundamental theorem for levered firm with its tax advantage

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathsf{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathsf{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right] + \mathsf{E}_t^Q\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathsf{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathsf{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \mathrm{l}_t \widetilde{V}_t^{\mathrm{l}}}{1+r_f} = \frac{\mathsf{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

Using rule 2 (linearity)

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right] + \mathrm{E}_t^Q\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \mathrm{l}_t \widetilde{V}_t^{\mathrm{l}}}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

# Proof (part I)



Rearranging terms

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right] + \mathsf{E}^Q\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t - \frac{\tau r_f \mathrm{l}_t \widetilde{V}^{\mathrm{l}}_t}{1+r_f} = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f}$$

Fundamental theorem for debtors

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right] + \mathrm{E}_t^Q\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \mathrm{l}_t \widetilde{V}_t^{\mathrm{l}}}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

# Proof (part I) 

Rearranging terms

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right] + \mathsf{E}^Q_t\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t - \frac{\tau r_f \mathrm{l}_t \widetilde{V}^{\mathrm{l}}_t}{1+r_f} = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f}$$

# Proof (part I)



Employing the definition of leverage ratio

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right] + \mathrm{E}_t^Q\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f l_t \widetilde{V}_t^{\mathrm{l}}}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

# Proof (part I)

Rearranging terms

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right] + \mathrm{E}_t^Q\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

$$\widetilde{V}_t^{\mathrm{l}} - \frac{\tau r_f \mathrm{l}_t \widetilde{V}_t^{\mathrm{l}}}{1+r_f} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{1+r_f}$$

Rearranging terms

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathrm{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathrm{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right] + \mathrm{E}^Q_t\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathrm{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t - \frac{\tau r_f \widetilde{D}_t}{1+r_f} = \frac{\mathrm{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f}$$

$$\widetilde{V}^{\mathrm{l}}_t \left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right) = \frac{\mathrm{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f}$$

**And finally have a nice recursive equation!**

$$
\begin{aligned}
\widetilde{V}^{\mathrm{l}}_t &= \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1} + \tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f} \\
\widetilde{V}^{\mathrm{l}}_t &= \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right] + \mathsf{E}^Q_t\left[\tau(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t)\right]}{1+r_f} \\
\widetilde{V}^{\mathrm{l}}_t &= \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f} + \frac{\tau r_f \widetilde{D}_t}{1+r_f} \\
\widetilde{V}^{\mathrm{l}}_t - \frac{\tau r_f \widetilde{D}_t}{1+r_f} &= \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{1+r_f} \\
\widetilde{V}^{\mathrm{l}}_t &= \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}.
\end{aligned}
$$

# Proof (detour)



We start with the recursive equation

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)(1+r_f)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)\left(1+k^{E,\mathrm{u}}\right)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right)}$$

$$\left(1 - \frac{\tau r_f l_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\widetilde{V}_t^{\mathrm{l}}}$$

$$\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = 1 + WACC_t$$

# Proof (detour)



Since cash flows of the unlevered firm are martingale-like, cost of capital can be used as discount rates

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)(1+r_f)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)\left(1+k^{E,\mathrm{u}}\right)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right)}$$

$$\left(1 - \frac{\tau r_f l_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\widetilde{V}_t^{\mathrm{l}}}$$

$$\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = 1 + WACC_t$$

# Proof (detour)



This $\widetilde{V}_t^{\mathrm{l}}$ must satisfy the following recursion equation – detour finished!

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)(1+r_f)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)\left(1+k^{E,\mathrm{u}}\right)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right)}$$

$$\left(1 - \frac{\tau r_f l_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\widetilde{V}_t^{\mathrm{l}}}$$

$$\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = 1 + WACC_t$$

# Proof (detour)



Rearranging terms

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathrm{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}$$

$$\widetilde{V}^{\mathrm{l}}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^{\mathrm{u}}_s\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_t\right)(1+r_f)^{s-t}}$$

$$\widetilde{V}^{\mathrm{l}}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}^{\mathrm{u}}_s\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_t\right)\left(1+k^{E,\mathrm{u}}\right)^{s-t}}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathrm{E}_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right)}$$

$$\left(1 - \frac{\tau r_f l_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = \frac{\mathrm{E}_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\widetilde{V}^{\mathrm{l}}_t}$$

$$\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = 1 + WACC_t$$

# Proof (detour) 

Using the definition of WACC type 2

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)(1+r_f)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\mathrm{l}_t\right)\left(1+k^{E,\mathrm{u}}\right)^{s-t}}$$

$$\widetilde{V}_t^{\mathrm{l}} = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right)}$$

$$\left(1 - \frac{\tau r_f l_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^{\mathrm{l}} + \widetilde{CF}_{t+1}^{\mathrm{u}}\right]}{\widetilde{V}_t^{\mathrm{l}}}$$

$$\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = 1 + WACC_t$$

q.e.d.

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}^Q_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+r_f)}$$

$$\widetilde{V}^{\mathrm{l}}_t = \sum_{s=t+1}^{T} \frac{\mathsf{E}^Q_t\left[\widetilde{CF}^{\mathrm{u}}_s\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_t\right)(1+r_f)^{s-t}}$$

$$\widetilde{V}^{\mathrm{l}}_t = \sum_{s=t+1}^{T} \frac{\mathsf{E}_t\left[\widetilde{CF}^{\mathrm{u}}_s\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_{s-1}\right)\cdots\left(1 - \frac{\tau r_f}{1+r_f}\,\mathrm{l}_t\right)\left(1+k^{E,\mathrm{u}}\right)^{s-t}}$$

$$\widetilde{V}^{\mathrm{l}}_t = \frac{\mathsf{E}_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)(1+k^{E,\mathrm{u}})}$$

$$\left(1 - \frac{\tau r_f l_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = \frac{\mathsf{E}_t\left[\widetilde{V}^{\mathrm{l}}_{t+1} + \widetilde{CF}^{\mathrm{u}}_{t+1}\right]}{\widetilde{V}^{\mathrm{l}}_t}$$

$$\left(1 - \frac{\tau r_f \mathrm{l}_t}{1+r_f}\right)\left(1+k^{E,\mathrm{u}}\right) = 1 + WACC_t.$$

Consider financing based on market values

$$\mathrm{l}_0 = 55\%, \qquad \mathrm{l}_1 = 10\%, \qquad \mathrm{l}_2 = 10\%.$$

WACC results from Miles-Ezzell adjustment,

$$\begin{aligned} WACC_0 &= \left(1 + k^{E,\mathrm{u}}\right)\left(1 - \frac{\tau r_f}{1 + r_f}\,\mathrm{l}_0\right) - 1 \\ &= (1 + 0.2)\left(1 - \frac{0.5 \times 0.1}{1 + 0.1} \times 0.55\right) - 1 = 17\% \\ WACC_1 &\approx 19.45\% \\ WACC_2 &\approx 19.45\%. \end{aligned}$$

The firm value is

$$V_0^{\mathrm{l}} = \frac{\mathsf{E}\left[\widetilde{CF}_1^{\mathrm{u}}\right]}{1 + WACC_0} + \frac{\mathsf{E}\left[\widetilde{CF}_2^{\mathrm{u}}\right]}{(1 + WACC_0)(1 + WACC_1)} + \frac{\mathsf{E}\left[\widetilde{CF}_3^{\mathrm{u}}\right]}{(1 + WACC_0)(1 + WACC_1)(1 + WACC_2)}$$

$$\approx \frac{100}{1.17} + \frac{110}{1.17 \times 1.1945} + \frac{121}{1.17 \times 1.1945 \times 1.1945} \approx 236.65.$$

Let $r_f = 10\%$. The leverage ratio is $l = 20\%$ and constant through lifetime.

With Miles-Ezzell adjustment the WACC is

$$WACC = (1 + k^{E,\mathrm{u}}) \left(1 - \frac{\tau r_f}{1 + r_f} l\right) - 1 \approx 18.9091\%$$

and the firm value is

$$\begin{aligned} V_0^{\mathrm{l}} &= \sum_{t=1}^{\infty} \frac{\mathrm{E}_0\left[\widetilde{CF}_t^{\mathrm{u}}\right]}{(1 + WACC)^t} \\ &= \sum_{t=1}^{\infty} \frac{FCF_0^u}{(1 + WACC)^t} = \frac{FCF_0^u}{WACC} \\ &\approx \frac{100}{0.189091} \approx 528.846. \end{aligned}$$

# Summary



The connection between costs of capital of a levered firm and an unlevered firm are given by an "adjustment formula".

In particular, one can derive a relation between $WACC$ and $k^{E,\mathrm{u}}$, so $WACC$ is indeed useful.

But this requires financing based on market values!