DCF: Firm income tax
Insolvency in the binomial model, II

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Two examples

We will consider the binomial model.

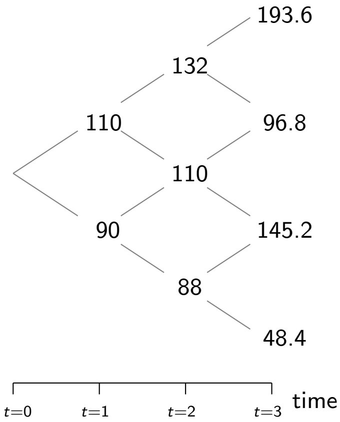
Taxes will be ignored again for simplicity (hence, $\widetilde{V}^{\mathrm{l}} = \widetilde{V}^{\mathrm{u}}$ and $\widetilde{CF}^{\mathrm{u}} = \widetilde{CF}^{\mathrm{l}}$).[1]

We consider a finite and an infinite example.

[1] Only index *u* from now on.

Finite example with insolvency

We start with our well-known example, have $k = 20\%$



levered cash flows $\widetilde{CF}^l$

and we add leverage

$D_0 = 140,\ D_1 = 62,\ D_2 = 0.$

At time $t = 2$ the company is unlevered.

For simplicity, we ignore taxes. They add complexity without adding insight.

From the lecture on "Cost of capital" we know the value at time $t = 0$,

$$
\begin{aligned}
V_0^l &= \frac{\mathrm{E}\left[\widetilde{CF}_1^l\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_2^l\right]}{(1+k)^2} + \frac{\mathrm{E}\left[\widetilde{CF}^l\right]}{(1+k)^3} \\
&= \frac{100}{1.2} + \frac{110}{1.2^2} + \frac{121}{1.2^3} \approx \underbrace{229.75}_{\text{firm value } V} > \underbrace{140}_{\text{debt value } D}
\end{aligned}
$$

Hence, the company is not over-indebted at $t = 0$.

Finite example with insolvency (cont.) 5

From the lecture on "Cost of capital" we know the value at time $t = 1$,

$$\widetilde{V}_1 \approx \begin{cases} 193.26, & \text{up} \\ 158.13, & \text{down} \end{cases} > 62 = D_1$$

Hence, the company is also not over-indebted at $t = 1$.

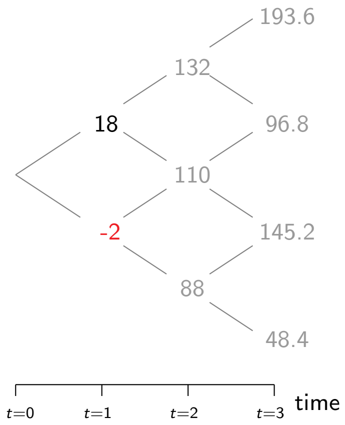
Finite example with insolvency (cont.)

But what about illiquidity? We start with $c = r_f$ and evaluate the cash flows to owners, for example at $t = 1$ given by

$$\widetilde{CF}_1^{l} - (1+c) \times D_0 + D_1 = \begin{cases} 110 - (1+10\%) \cdot 140 + 62 & \text{if up} \\ 90 - (1+10\%) \cdot 140 + 62 & \text{if down} \end{cases}$$

and get

Finite example with insolvency (cont.)



owner's cash flows

at state $\omega = d$ there is insolvency triggered by illiquidity.

Finally, let us evaluate the coupon rate for the "good" (non-insolvent) state $\omega = u$.

The creditors use the fundamental theorem. The fair value of their "investment" is $D_0 = 140$. What do they get in the case of complete sale of assets?

$$D_0 = \frac{\mathrm{E}_0^Q[\text{payments to creditors in } t=1]}{1+r_f} \qquad \text{by fundam. theorem}$$

$$140 = \frac{\overbrace{(1+c)\times 140}^{\text{no insolvency in } u}\times 0.0833 + \overbrace{(158.12+90)}^{\text{compl. sale in } d}\times 0.9167}{1+10\%} \qquad \text{from}^2$$

$$c \approx -729.9\%$$

[2] The Q-probabilities are from lecture "Conclusions martingale-like CF". $158.12 + 90 = V_1(d) + CF_1(d)$ and V_1 is from lecture "Cost of capital..."

Determining a

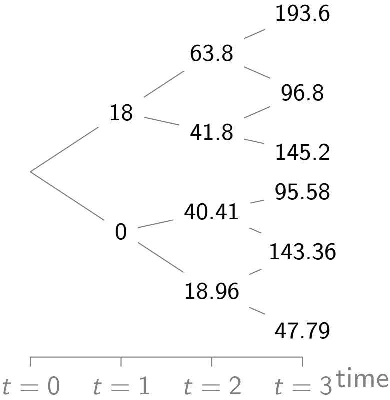
We already know the answer: Solution is partial sale of assets $(c = r_f)$ and $a < 1$.

Fundamental theorem

$$D_0 = \frac{(1 + r_f)\, D_0\, Q(u) + \left(\widetilde{CF}_1^{\mathrm{u}}(d) + \widetilde{D}_1(d) + a\widetilde{V}_1^{\mathrm{u}}(d)\right) Q(d)}{1 + r_f}$$

$$140 = \frac{(1 + 10\%) \times 140 \times 0.0833 + \left(90 + 62 + a \times 158.12\right) \times 0.9167}{1 + 10\%}$$

$$a = \frac{2}{158.13} \approx 1.26\%$$



Sale of

$$a = \frac{2}{158.13} \approx 1.26\%$$

of assets leads to liquidity.

Remark: $D_1 = 64$ would also solve the issue...

owner's cash flows after sales of assets (gross cash flow minus $(1 + r_f)D_{t-1}$ plus D_{t+1} plus asset sales)

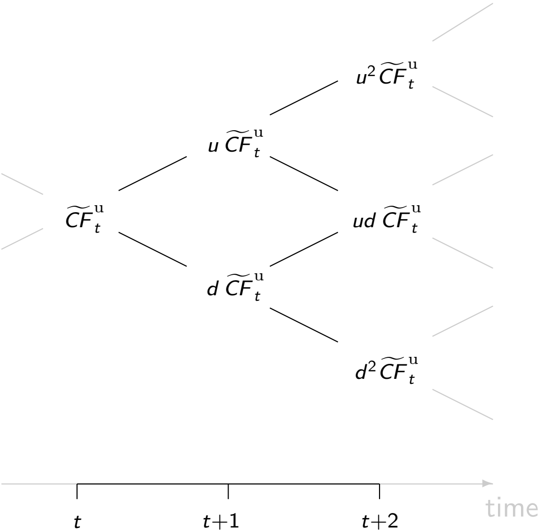
We did not change the financing policy.

Hence, valuation ignoring insolvency and valuation taking insolvency into account give the same value at $t = 0$.

This is not the case for values in the future ($t > 0$, in the down states) since we sold assets. $\widetilde{V}_1^{\mathrm{u}}, \widetilde{V}_2^{\mathrm{u}}$ are different.

Infinite example: Cash flow development 12

The model will be based on the infinite example, $g = 0$[3], ex ante $\widetilde{CF}^{\mathrm{u}}$ given



[3] See lecture "Information and conditional expectation".

The model is time-homogeneous: we focus on one period from t to $t+1$ and we assume $l = \frac{\widetilde{D}_t}{\widetilde{V}_t^{\mathrm{u}}}$ constant.[4]

Remark: In this condition $\widetilde{V}_t^{\mathrm{u}}$ is ex post value. So if assets are sold ($a_t > 0$), debt $\widetilde{D}_t$ decreases!

We define two thresholds

$$l^* := \frac{dk^{E,\mathrm{u}}}{1 + r_f - d}$$
$$l^{**} := \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} d.$$

using $r_f, k^{E,\mathrm{u}}, d$ from infinite example.

[4] This is "financing based on market values".

$l^* < l^{**}$ indeed

Remark: We have $l^* < l^{**}$.

$$
\begin{aligned}
d &< \frac{1 + r_f}{1 + k^{E,\mathrm{u}}} && \text{from arbitrage} \\
0 &< 1 + r_f - d(1 + k^{E,\mathrm{u}}) \\
k^{E,\mathrm{u}}(1 + r_f) &< (1 + k^{E,\mathrm{u}})(1 + r_f - d) \\
\frac{k^{E,\mathrm{u}}}{1 + r_f - d} &< \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} \\
\frac{dk^{E,\mathrm{u}}}{1 + r_f - d} &< \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} d \\
l^* &< l^{**}
\end{aligned}
$$

Two types of insolvency

We have three cases

$$\begin{aligned} I \leq I^* &\quad \text{no insolvency} \\ I^* < I < I^{**} &\quad \text{insolvency, and } c = r_f \text{ and } a \in (0,1) \\ I^{**} \leq I &\quad \text{insolvency, and } c > r_f \text{ and } a = 1. \end{aligned}$$

The case $l^* < l < l^{**}$

Checking illiquidity (with ex ante values!)

$$
\begin{aligned}
\widetilde{CF}^{\mathrm{l}}_{t+1}(d) - \left((1+r_f)\widetilde{D}_t - \widetilde{D}_{t+1}(d)\right) \\
&= k^{E,\mathrm{u}}\widetilde{V}^{\mathrm{u}}_t\, d - \left((1+r_f)l\widetilde{V}^{\mathrm{u}}_t - l\widetilde{V}^{\mathrm{u}}_t\, d\right) \quad \text{Gordon-Shapiro, down} \\
&= \left(dk^{E,\mathrm{u}} - l(1+r_f-d)\right)\widetilde{V}^{\mathrm{u}}_t
\end{aligned}
$$

and $dk^{E,\mathrm{u}} < l(1+r_f-d)$ if and only if $l^* < l$.

Evaluating the share a

Using the leverage ratio l and the share a

$$
\begin{aligned}
0 &= \widetilde{CF}^{\mathrm{l}}_{t+1}(d) - (1+r_f)\widetilde{D}_t + \widetilde{D}_{t+1}(d) + a\,\overbrace{\widetilde{V}^{\mathrm{u}}_{t+1}(d)}^{\text{ex ante}} \\
&= k^{E,\mathrm{u}}\,\widetilde{V}^{\mathrm{u}}_t\,d - (1+r_f)l\widetilde{V}^{\mathrm{u}}_t + ld\,\overbrace{(1-a)\widetilde{V}^{\mathrm{u}}_t}^{\text{ex post}} + ad\,\widetilde{V}^{\mathrm{u}}_t \\
&= \widetilde{V}^{\mathrm{u}}_t\left(k^{E,\mathrm{u}}\,d - (1+r_f)l + (1-a)ld + ad\right) \\
a &= \frac{\left(\frac{1+r_f}{d} - 1\right)l - k^{E,\mathrm{u}}}{1-l}
\end{aligned}
$$

The value is meaningful only if $a \in (0,1)$.

Ratio a can be rearranged to $a = \frac{k^{E,\mathrm{u}}}{l^*}\frac{l-l^*}{1-l}$. Obviously positive, $a > 0$.

Also less then 1, if $l < l^{**}$:

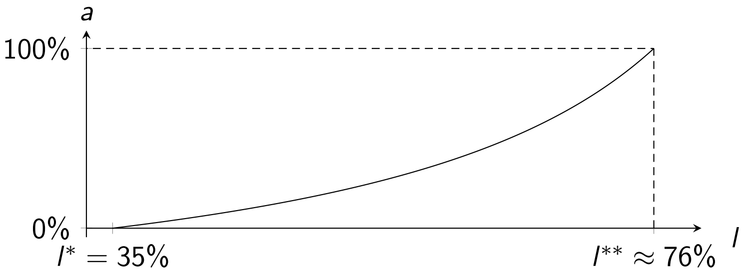
$$l < l^{**} = \frac{1+k^{E,\mathrm{u}}}{1+r_f} d$$

$$\frac{1+r_f}{d} l < 1 + k^{E,\mathrm{u}}$$

$$a = \frac{\left(\frac{1+r_f}{d} - 1\right) l - k^{E,\mathrm{u}}}{1-l} < 1$$

A numerical example: share

Assume $k^{E,\mathrm{u}} = 20\%$, $r_f = 10\%$ and $d = 0.7$. Then



We have

$$
\begin{aligned}
(1+r_f)\widetilde{D}_t &= (1+c_t)\,\widetilde{D}_t\,Q(u) + \left(\widetilde{CF}^{\mathrm{u}}_{t+1}(d) + \widetilde{V}^{\mathrm{u}}_{t+1}(d)\right)Q(d)\\
(1+r_f)l\widetilde{V}^{\mathrm{u}}_t &= (1+c_t)\,l\widetilde{V}^{\mathrm{u}}_t\,Q(u) + (1+k^{E,\mathrm{u}})d\widetilde{V}^{\mathrm{u}}_t\,Q(d)\\
1+c_t &= (1+k^{E,\mathrm{u}})\left(u - \frac{1-l}{l}\frac{Q(d)}{Q(u)}d\right)\,.
\end{aligned}
$$

Notice

$$1+c \le (1+k^{E,\mathrm{u}})u\,.$$

The coupon is larger then r_f:

$$l > l^{**} = \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} d$$

$$(1 + r_f) l \widetilde{V}_t^{\mathrm{u}} > \left(1 + k^{E,\mathrm{u}}\right) d \widetilde{V}_t^{\mathrm{u}}$$

$$(1 + r_f) \widetilde{D}_t Q(d) > \left(\widetilde{CF}_{t+1}^{\mathrm{u}}(d) + \widetilde{V}_{t+1}^{\mathrm{u}}(d)\right) Q(d)$$

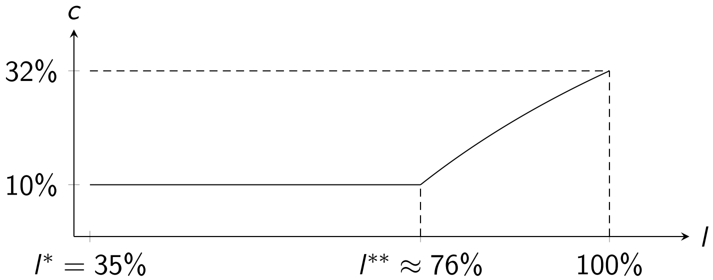
$$(1 + r_f) \widetilde{D}_t > (1 + r_f)\, \widetilde{D}_t\, Q(u) + \left(\widetilde{CF}_{t+1}^{\mathrm{u}}(d) + \widetilde{V}_{t+1}^{\mathrm{u}}(d)\right) Q(d)$$

from equation (4) in “Insolvency binomial model”

$$(1 + r_f) \widetilde{D}_t = (1 + c_t)\, \widetilde{D}_t\, Q(u) + \left(\widetilde{CF}_{t+1}^{\mathrm{u}}(d) + \widetilde{V}_{t+1}^{\mathrm{u}}(d)\right) Q(d)$$

Q.E.D.

Assume $k^{E,\mathrm{u}} = 20\%$, $r_f = 10\%$, $u = 1.1$ and $d = 0.7$. Then



Out of interest, let's compute the (ex post) cash flows to the owners.

We start with a hypothetical ex ante cash flow of 20 at date 0 and with $u = 1.1$ (up) and $d = 0.7$ (down). Assume $l = 50\%$.

Cash flows following a down move are easy to determine: the firm becomes illiquid. By selling a fraction

$$a = \frac{\left(\frac{1+r_f}{d} - 1\right) l - k^{E,\mathrm{u}}}{1 - l} \approx 17.14\%$$

of the firm's assets the net cash flows to the owners are zero.

For example, at $t = 1$ in the up state:

$$\overbrace{20}^{=CF_0^u} \cdot \overbrace{1.1}^{=u} - \overbrace{(1+10\%)}^{=1+r_f} \cdot \overbrace{50\%}^{=l} \cdot \overbrace{100}^{=V_0^u=\frac{CF_0^u}{k^{E,\mathrm{u}}}} + \overbrace{50\%}^{l} \cdot \overbrace{110}^{=\widetilde{V}_1^{\mathrm{u}}=uV_0^u} = 22$$

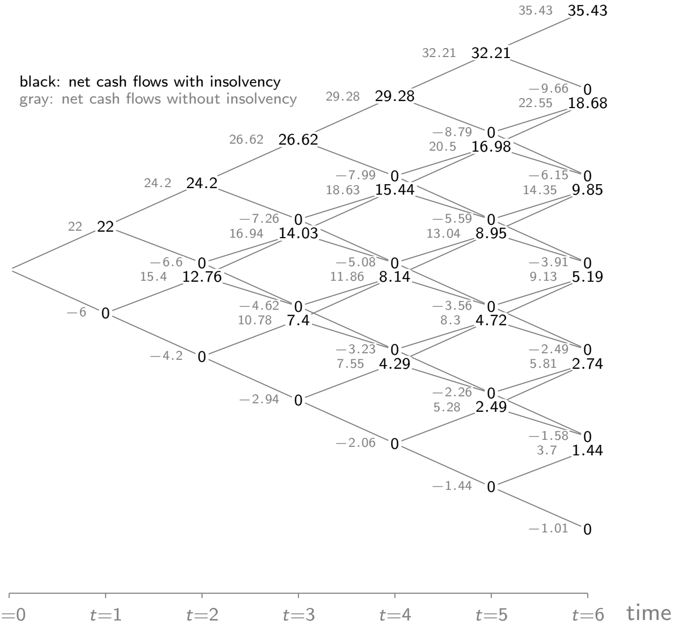
and in the down state

$$\overbrace{20}^{=CF_0^u} \cdot \overbrace{0.7}^{=d} - \overbrace{(1+10\%)}^{=1+r_f} \cdot \overbrace{50\%}^{=l} \cdot \overbrace{100}^{=V_0^u=\frac{CF_0^u}{k^{E,\mathrm{u}}}} + \overbrace{50\%}^{l} \cdot \overbrace{70}^{=\widetilde{V}_1^{\mathrm{u}}=dV_0^u} = -6$$

etc.

Choose a such that no illiquidity in the down state at $t+1$
The net cash flow in the up state at $t+1$ using the same rules as above results in

Cash flows to owners



Insolvency in the binomial model has the following features:

- Illiquidity is avoided by selling (part of) the firm's assets.
- Asset sales reduce future cash flows proportionally.
- In the infinite-horizon case, we distinguish between 1) debt is still rolled over at the risk-free rate (only a partial asset sale) or 2) firm no longer belongs to the original owners (complete asset sale).
- The resulting net cash flows can be evaluated, the resulting tree is no longer recombining.