DCF: Firm income tax
Insolvency: Infinite example

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Insolvency in the infinite example

We want to establish a model of the following form

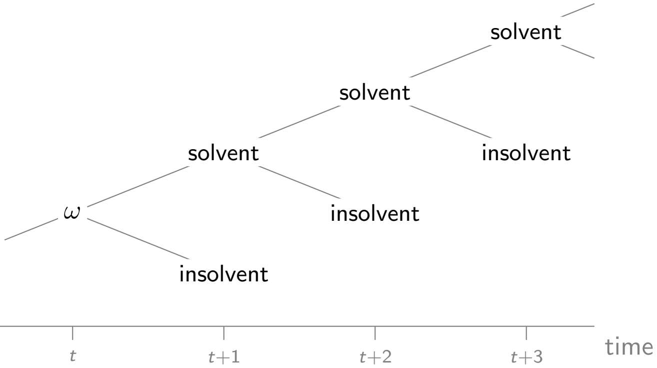
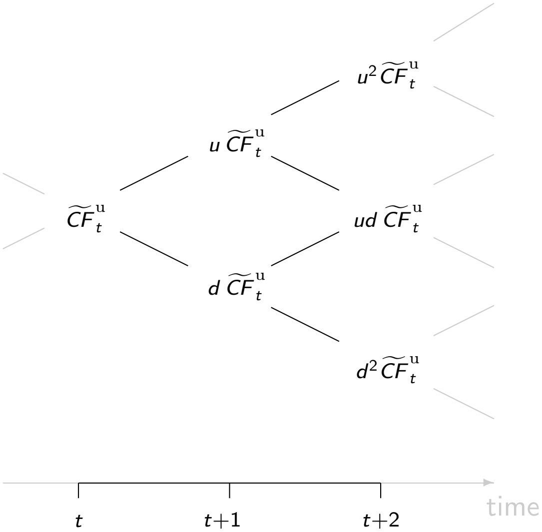


Figure: Insolvency in each down state.

Cash flow development

The model will be based on the infinite example, $g = 0$[1], $\widetilde{CF}^{\mathrm{u}}$ given



[1] See lecture "Information and conditional expectation".

Taxes will be ignored again for simplicity (hence, $\widetilde{V}^{\mathrm{l}} = \widetilde{V}^{\mathrm{u}}$ and $\widetilde{CF}^{\mathrm{u}} = \widetilde{CF}^{\mathrm{l}}$).[2]

This means:

1. Since in the infinite model without debt $\widetilde{V}_t^{\mathrm{u}} = \frac{\widetilde{CF}_t^{\mathrm{u}}}{k^{E,\mathrm{u}}}$, neither illiquidity nor over-indebtedness can occur
   $\Longrightarrow$ Debt (only) causes insolvency!

2. The business model $(\widetilde{CF}^{\mathrm{u}}, k^{E,\mathrm{u}})$ is not changed due to insolvency; only ownership changes.
   After the insolvent states the company still exists, but is not owned by the shareholders anymore.

[2] Only index u from now on.

Outline

This model is time-homogeneous, so it suffices to focus on one period (from t to $t+1$).

1. We determine leverage $l(u) = \frac{\widetilde{D}(u)}{\widetilde{V}^{\mathrm{u}}(u)}$, in a *solvent state*. By time-homogeneity, this also pins down debt in any solvent state.
2. Based on this, we deduce the coupon rate c_t.
3. We determine $l(d) = \frac{\widetilde{D}(d)}{\widetilde{V}^{\mathrm{u}}(d)}$, in the *insolvent state*.
4. We then make sure that the (levered) firm is over-indebted and illiquid!

Coupon rate c—idea

Cost of capital is k, risk-free rate r_f.

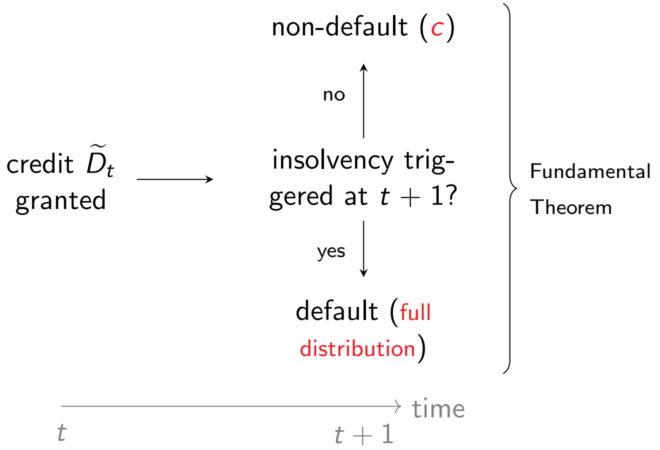


Figure: Remember from last lecture?

A threshold

We define

$$l^* := \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} d.$$

using $r_f, k^{E,\mathrm{u}}, d$ from infinite example.

Coupon rate

Again we use the fundamental theorem

$$\widetilde{D}_t = \frac{\mathrm{E}_t^Q[\text{payments to creditors in } t+1]}{1+r_f}$$

Payments in $t+1$ can be

solvent state (up) Creditors get $(1+c_t)\widetilde{D}_t$.

insolvent state (down) Complete transfer of assets, creditors get $\widetilde{V}_{t+1}^{\mathrm{u}} + \widetilde{CF}_{t+1}^{\mathrm{u}}$.

This sum can be simplified using Gordon[3] and growth rate d so that creditors get $\widetilde{V}_{t+1}^{\mathrm{u}} + \widetilde{CF}_{t+1}^{\mathrm{u}}$.

[3] See lecture "Conclusions from martingale..."

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[3]See lecture "Conclusions from martingale. . . "

Finally we get

$$\widetilde{D}_t = \frac{Q(u)(1+c_t)\widetilde{D}_t + Q(d)\widetilde{V}_t^{\mathrm{u}}(1+k^{E,\mathrm{u}})d}{1+r_f}$$

$$1+r_f = Q(u)(1+c_t) + Q(d) \underbrace{\frac{\widetilde{V}_t^{\mathrm{u}}}{\widetilde{D}_t}}_{=:l^{-1}(u)} (1+k^{E,\mathrm{u}})d \qquad \text{see }^4$$

This equation determines the coupon rate c_t[8] if the ratio $l(u)$ is given!

Remark: in $l(u)$ we use unlevered firm value. Because taxes are absent, this is levered firm value and $l(u)$ is leverage ratio in up-state.

[7] $Q(u), Q(d)$ were evaluated in lecture "Conclusions from martingale..."

[8] Now without index t.

Three remarks:

1. The coupon does not depend on $l(d)$.
2. The last equations shows

$$c = A + \frac{B}{l(u)}$$

   (with numbers A and B): Coupon depends on inverse of leverage ratio in the solvent state.
3. Do we have $c > r_f$? Otherwise, the model would be inconsistent!

The coupon rate $c > r_f$?

Is this coupon greater than the risk-free rate? Yes:

$$I(u) \geq I^*$$

The coupon rate $c > r_f$?

Is this coupon greater than the risk-free rate? Yes:

$$l(u) \geq \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} d.$$

The coupon rate $c > r_f$?

Is this coupon greater than the risk-free rate? Yes:

$$l(u)(1 + r_f) \geq (1 + k^{E,\mathrm{u}})d.$$

The coupon rate $c > r_f$?

Is this coupon greater than the risk-free rate? Yes:

$$(1 + r_f) Q(d) \geq (1 + k^{E,\mathrm{u}}) l^{-1}(u) Q(d) d.$$

The coupon rate $c > r_f$?

Is this coupon greater than the risk-free rate? Yes:

$$(1+c)Q(u)+(1+r_f)Q(d) \geq \underbrace{(1+c)Q(u)+(1+k^{E,\mathrm{u}})l^{-1}(u)Q(d)d}_{=1+r_f \text{ by slide above}}.$$

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The coupon rate $c > r_f$?

Is this coupon greater than the risk-free rate? Yes:

$$c \geq r_f$$

Our company is over-indebted in down-state if equity is negative,

$$\begin{aligned}\widetilde{V}^{\mathrm{u}}_{t+1}(d) - \widetilde{D}_{t+1}(d) &= \widetilde{V}^{\mathrm{u}}_{t}(u)\, d - \widetilde{D}_{t+1}(d) \\ &= \widetilde{V}^{\mathrm{u}}_{t}(u)\, d - l(d)\widetilde{V}^{\mathrm{u}}_{t+1}(d) \\ &= \widetilde{V}^{\mathrm{u}}_{t}(u)\, d - l(d)d\,\widetilde{V}^{\mathrm{u}}_{t}(u)\end{aligned}$$

which is the case if

$$1 < l(d)\,.$$

Our company is illiquid in down-state if cash flow to owners is negative. The cash flow is given by

$$\begin{aligned}
\widetilde{CF}^{\mathrm{u}}_{t+1}(d) + \widetilde{D}_{t+1}(d) - (1+c)\widetilde{D}_t(u) \\
&= \widetilde{CF}^{\mathrm{u}}_{t+1}(d) + l(d)\widetilde{V}^{\mathrm{u}}_{t+1}(d) - (1+c)l(u)\widetilde{V}^{\mathrm{u}}_t(u) \\
&= k^{E,\mathrm{u}}\widetilde{V}^{\mathrm{u}}_{t+1}(d) + l(d)d\widetilde{V}^{\mathrm{u}}_t(u) - (1+c)l(u)\widetilde{V}^{\mathrm{u}}_t(u) \\
&= k^{E,\mathrm{u}}d\widetilde{V}^{\mathrm{u}}_t(u) + l(d)d\widetilde{V}^{\mathrm{u}}_t(u) - (1+c)l(u)\widetilde{V}^{\mathrm{u}}_t(u)
\end{aligned}$$

and this is negative if

$$k^{E,\mathrm{u}}d + l(d)d - (1+c)l(u) < 0$$

or

$$l(d) < l(u)\frac{1+c}{d} - k^{E,\mathrm{u}} .$$

Finally, we need[9]

$$
\begin{aligned}
l^* &\leq l(u) \\
1 < l(d) &< l(u)\frac{1+c}{d} - k^{E,\mathrm{u}}
\end{aligned}
$$

for both triggers to be met.

We have finally built a model with infinite time (based on the infinite example) where in every down-state there is insolvency. We even have a complete transfer of assets in default.

[9] That the interval for $l(d)$ is not empty follows from $l(u) > l^*$.

This is from problem 2.10 from the book.

Use $u = 1.1$, $d = 0.7$, $k^{E,\mathrm{u}} = 20\%$, and $r_f = 10\%$ gives

$$l^* = \frac{1 + k^{E,\mathrm{u}}}{1 + r_f} d \approx 76.36\% ,$$

and

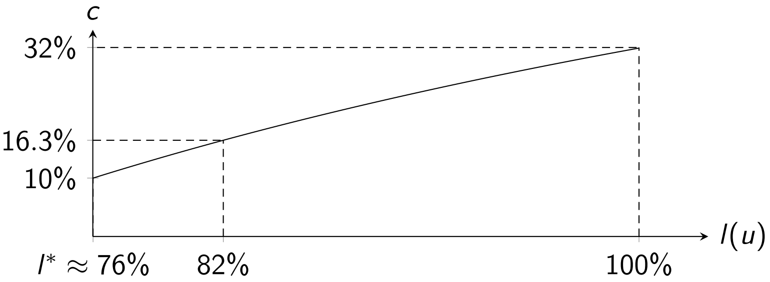

Figure: Coupon rate.

If $l(u) = 82\%$ then we have $c \approx 16.39\%$.

We get insolvency if

$$\underbrace{1 <}_{\text{over-indebt.}} \quad l(d) \quad \underbrace{< l(u)\frac{1+c}{d} - k^{E,\mathrm{u}} \approx 1.1995}_{\text{illiquidity}}$$

Infinite example: up is solvent state, down is insolvent state.

The coupon c can be determined and depends linearly on $l^{-1}(u)$.

Specific $l(d)$ insures that both triggers are met.