DCF: Basic Concepts
Fundamental theorem

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Fundamental theorem 2

Remember

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + k_t}.$$

Another way to handle risk are risk-neutral probabilities Q.

$$\widetilde{V}_t = \frac{\mathrm{E}_t^{Q}\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + r_f}.$$

Does Q always exist?
This is not a trivial question! For example, solutions of the form $q(state) = -1$ or $q(state) = 2$ would not be considered as 'probabilities'.

Theorem (fundamental theorem) *If the markets are free of arbitrage, there is a probability Q such that for* **all** *claims*

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q \left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1} \right]}{1 + r_f}.$$

Holds for any claims!

What about a proof? Forget it.

How to get Q for valuation of firms? No idea.

So why is this helpful? We will see (much) later.

Is there at least an interpretation of Q? Yes!

Risk-neutrality of Q

Why do we call Q a risk-neutral probability?

Look at the following:

$$1 + r_f = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{\widetilde{V}_t} \qquad \text{fundamental theorem}$$

$$1 + r_f = \mathrm{E}_t^Q\left[\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t}\right] \qquad \text{rule 5}$$

$$r_f = \mathrm{E}_t^Q\left[\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t}\right] - \mathrm{E}_t^Q[1] \qquad \text{rule 3}$$

$$r_f = \mathrm{E}_t^Q\Bigg[\underbrace{\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t} - 1}_{\text{return on holding a share}}\Bigg] \qquad \text{rule 2.}$$

The last equation

$$r_f = \mathrm{E}_t^Q[\text{return}]$$

simply says: if we change our probabilities to Q, any security has expected return r_f.

Or: **the world is risk-neutral under** Q.

Uniqueness of Q

Is Q unique? Or can the value of the company depend on Q?

If the cash flows of the firm can be duplicated by traded assets **(market is complete)** then any Q will lead to the same value.

Proof? Forget this as well...

Two assumptions

From now on two assumptions will always hold:

Assumption *The markets are free of arbitrage.*

Assumption *The cash flows of the firm can be duplicated by traded assets.*

$\Longrightarrow$ The risk-neutral probability Q exists and is (in some sense) unique.

Theorem *The market value of the firm is (also) given by*

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}]}{1+r_f} + \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+2}]}{(1+r_f)^2} + \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+3}]}{(1+r_f)^3} + \dots + \frac{\mathrm{E}_t^Q[\widetilde{CF}_T]}{(1+r_f)^{T-t-1}}$$

We are now proving this theorem using our rules.

Reformulate our fundamental theorem

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1+r_f}.$$

Using the fundamental theorem next period

$$\widetilde{V}_{t+1} = \frac{\mathrm{E}_{t+1}^Q\left[\widetilde{CF}_{t+2} + \widetilde{V}_{t+2}\right]}{1+r_f},$$

the result is

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \frac{\mathrm{E}_{t+1}^Q\left[\widetilde{CF}_{t+2}+\widetilde{V}_{t+2}\right]}{1+r_f}\right]}{1+r_f}.$$

Because the risk-free rate is deterministic, we can apply the Linearity Rule (no. 3) of conditional expectation and get

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}] + \frac{\mathrm{E}_t^Q\left[\mathrm{E}_{t+1}^Q[\widetilde{CF}_{t+2}+\widetilde{V}_{t+2}]\right]}{1+r_f}}{1+r_f}.$$

Using the Rule of Iterated Expectation (no. 4) we get

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}] + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+2}+\widetilde{V}_{t+2}\right]}{1+r_f}}{1+r_f}$$

Rearranging and using linearity again gives

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}]}{1+r_f} + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+2}\right]}{(1+r_f)^2} + \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+2}\right]}{(1+r_f)^2}$$

This procedure can be applied again and again: We see a pattern here!

If T is the last point in time where $\widetilde{V}_T = 0$, we get finally our first valuation equation.

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}]}{1+r_f} + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+2}\right]}{(1+r_f)^2} + \ldots + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_T\right]}{(1+r_f)^{T-t-1}}$$

This is what we wanted to show.

If markets are free of arbitrage, “some” Q exists. It looks like a probability (between 0 and 1) and can be interpreted as implying only “risk-neutral” returns. We call it risk-neutral probability.

If markets are complete, Q is unique.

Q may be intuitive, but it is not directly observable and is, at first glance, of limited use.

We can value every claim using Q and r_f.