DCF: Basic Concepts
Cost of capital and market value

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> the investor a more volatile consumption stream. (It is common instead to write prices as a discounted value using a risk-adjusted discount factor, e.g., $p_t^i = \sum_{j=1}^{\infty} E_t d_{t+j}^i / (R^i)^j$, but this approach is difficult to use correctly for multiperiod problems, especially when expected returns can vary over time.)
>
> At a deeper level, the expectation in the two period formula $p = E(mx)$

Taken from J. Cochrane, Asset Pricing (2nd ed.).

In modern asset pricing, the "cost of capital" is widely viewed as a problematic concept (the term itself appears only once in the book mentioned above). I would like to challenge this view.

What are cost of capital?

> The discount rate for a project is its cost of capital: The expected return of securities with comparable risk and horizon.

Taken from J. Berk, P. DeMarzo, Corporate Finance (4th ed.).

This (typical) definition contains two elements that we will distinguish:

1. Expected rate of return,
2. Discount rate.

(Sometimes "opportunity costs" or "yields" are mentioned.)

We will see that the two terms should not be treated as interchangeable.

First, let us ignore uncertainty.

Notation:

CF	firm's free cash flow
V	value of the firm

Idea:

Cost of capital is used for **discounting**, hence

$$V_0 = \frac{CF_1}{1+k_0} + \frac{CF_2}{(1+k_0)(1+k_1)} + \ldots$$

This idea shall also be applied in the future: at $t = 1$ we want to have

$$V_1 = \frac{CF_2}{1 + k_1} + \frac{CF_3}{(1 + k_1)(1 + k_2)} + \ldots$$

where k_1 is the **same cost of capital from the last slide!**

Cost of capital: a rough definition

Then the definition of cost of capital should run

$$\boxed{k_t =_{\text{Def}} \frac{CF_{t+1} + V_{t+1}}{V_t} - 1}$$

Implication: Costs of capital are inevitably (expected) returns.

A different approach could be

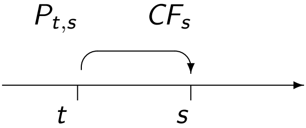
$$V_0 = \frac{CF_1}{1+k_0} + \frac{CF_2}{(1+k_1)^2} + \dots$$

$$\text{but then} \implies V_1 \stackrel{?}{=} \frac{CF_2}{1+k_1} + \frac{CF_3}{(1+k_2)^2} + \dots$$

Here the costs of capital are **yields**. We do not think much along this course (this is a different concept), because it is difficult to observe yields empirically.

Cost of capital: discount rates 8

You pay at time t a price $P_{t,s}$ for cash flow CF_s due at s:



We would then define a discount rate as

$$P_{t,s} =_{\text{Def}} \frac{CF_s}{(1+\kappa_{t,s})^{s-t}}$$

What relation exists between these discount rates and (expected) returns (=cost of capital)?

We get back to this...

Which definition is **useful for our purpose**? The following definition turns out to be appropriate

Definition

$$\widetilde{k}_t := \frac{\mathrm{E}_t[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}]}{\widetilde{V}_t} - 1.$$

- return from t to $t+1$
- conditional expectation!
- hence: cost of capital uncertain, random variable seen from today!

(The last point is probably not surprising for the practitioner.)

For example:

$$k =_{\text{Def?}} \frac{\mathrm{E}\left[\widetilde{V}_{t+1} + \widetilde{CF}_{t+1}\right]}{\mathrm{E}[\widetilde{V}_t]} - 1$$

cannot be rearranged to $\widetilde{V}_t = \ldots$ (and is not an expected return!).

The same applies to

$$k =_{\text{Def?}} \mathrm{E}\left[\frac{\widetilde{V}_{t+1} + \widetilde{CF}_{t+1}}{\widetilde{V}_t} - 1\right]$$

Rapp (zfbf, 2006) and Laitenberger (zfb, 2006) have analyzed that problem (in German).

But: One cannot discount using uncertain cost of capital.

Heroic Assumption
The cost of capital of a firm is assumed to be non-random (deterministic).

First valuation equation

Theorem *Then, the (future) market value of the firm is given by*

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}]}{1+k_t} + \frac{\mathrm{E}_t[\widetilde{CF}_{t+2}]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}_t[\widetilde{CF}_{t+3}]}{(1+k_t)(1+k_{t+1})(1+k_{t+2})} + \ldots + \frac{\mathrm{E}_t[\widetilde{CF}_T]}{(1+k_t)(1+k_{t+1})\cdots(1+k_{T-1})}$$

We are now proving this theorem using our rules.

Reformulate our definition

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + k_t}.$$

Using the appropriate relation next period

$$\widetilde{V}_{t+1} = \frac{\mathrm{E}_{t+1}\left[\widetilde{CF}_{t+2} + \widetilde{V}_{t+2}\right]}{1 + k_{t+1}},$$

the result is

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1} + \frac{\mathrm{E}_{t+1}\left[\widetilde{CF}_{t+2} + \widetilde{V}_{t+2}\right]}{1 + k_{t+1}}\right]}{1 + k_t}.$$

Because cost of capital is deterministic, we can apply the Linearity Rule (no. 3) of conditional expectation and get

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}] + \frac{\mathrm{E}_t\left[\mathrm{E}_{t+1}[\widetilde{CF}_{t+2}+\widetilde{V}_{t+2}]\right]}{1+k_{t+1}}}{1+k_t}.$$

Using the Rule of Iterated Expectation (no. 4) we get

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}] + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}+\widetilde{V}_{t+2}\right]}{1+k_{t+1}}}{1+k_t}$$

Rearranging and using Linearity again gives

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}]}{1+k_t} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}_t\left[\widetilde{V}_{t+2}\right]}{(1+k_t)(1+k_{t+1})}$$

This procedure can be applied again and again: Let us continue one more time.

We use

$$\widetilde{V}_{t+2} = \frac{\mathrm{E}_{t+2}\left[\widetilde{CF}_{t+3} + \widetilde{V}_{t+3}\right]}{1 + k_{t+2}},$$

and plug it into the last equation

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}]}{1 + k_t} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1 + k_t)(1 + k_{t+1})} + \frac{\mathrm{E}_t\left[\frac{\mathrm{E}_{t+2}\left[\widetilde{CF}_{t+3} + \widetilde{V}_{t+3}\right]}{1 + k_{t+2}}\right]}{(1 + k_t)(1 + k_{t+1})}$$

Again, k_{t+2} is deterministic and we apply Linearity

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}]}{1+k_t} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}_t\left[\mathrm{E}_{t+2}\left[\widetilde{CF}_{t+3}\right]\right]}{(1+k_t)(1+k_{t+1})(1+k_{t+2})} + \frac{\mathrm{E}_t\left[\mathrm{E}_{t+2}\left[\widetilde{V}_{t+3}\right]\right]}{(1+k_t)(1+k_{t+1})(1+k_{t+2})}$$

Using Iterated Expectation this implies

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}]}{1+k_t} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+3}\right]}{(1+k_t)(1+k_{t+1})(1+k_{t+2})} + \frac{\mathrm{E}_t\left[\widetilde{V}_{t+3}\right]}{(1+k_t)(1+k_{t+1})(1+k_{t+2})}$$

We see a pattern here!

Proof VI

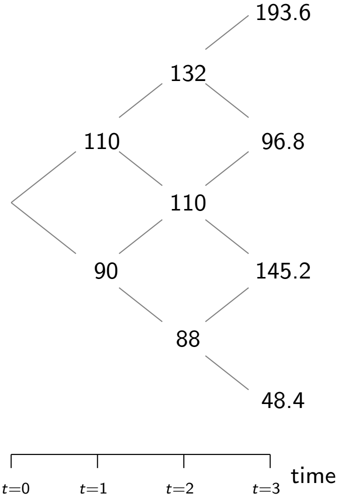
If T is the last point in time where $\widetilde{V}_T = 0$, we get finally our first valuation equation.

$$\widetilde{V}_t = \frac{\mathrm{E}_t[\widetilde{CF}_{t+1}]}{1+k_t} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1+k_t)(1+k_{t+1})} + \dots + \frac{\mathrm{E}_t\left[\widetilde{CF}_T\right]}{(1+k_t)(1+k_{t+1})\cdots(1+k_{T-1})}$$

This is what we wanted to show.

Finite example

Let us look again at the finite example



Assume $k = 20\%$.
The value today is given

$$V_0 = \frac{E\left[\widetilde{CF}_1\right]}{1+k} + \frac{E\left[\widetilde{CF}_2\right]}{(1+k)^2} + \frac{E\left[\widetilde{CF}_3\right]}{(1+k)^3}$$
$$= \frac{100}{1.2} + \frac{110}{1.2^2} + \frac{121}{1.2^3} \approx 229.75.$$

Although not immediately evident here, we determine market for time $t = 1$. But, this value is uncertain: we must take into account at which node (state) we are.

If up

$$\widetilde{V}_1(u) = \frac{\mathrm{E}\left[\widetilde{CF}_2(u\omega)\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_3(u\omega)\right]}{(1+k)^2} = \frac{121}{1.2} + \frac{133.1}{1.2^2} \approx 193.26\ ,$$

If down

$$\widetilde{V}_1(d) = \frac{\mathrm{E}\left[\widetilde{CF}_2(d\omega)\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_3(d\omega)\right]}{(1+k)^2} = \frac{99}{1.2} + \frac{108.9}{1.2^2} \approx 158.13$$

Altogether

$$\widetilde{V}_1 \approx \begin{cases} 193.26, & \text{if the development in } t = 1 \text{ is up,} \\ 158.13, & \text{if the development in } t = 1 \text{ is down.} \end{cases}$$

Also

$$\widetilde{V}_2 \approx \begin{cases} 121.00, & \text{if the development is up–up,} \\ 100.83, & \text{if the development is up–down or down-up,} \\ 80.67, & \text{if the development is down–down.} \end{cases}$$

Infinite example

Remember our **Theorem**

$$
\begin{aligned}
\widetilde{V}_t &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t[\widetilde{CF}_s]}{(1+k)^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{(1+g)^{s-t}\,\widetilde{CF}_t}{(1+k)^{s-t}} && \text{see video "Information"}, g = 0 \\
&= \widetilde{CF}_t \sum_{s=t+1}^{\infty} \frac{1}{(1+k)^{s-t}} \\
&= \frac{\widetilde{CF}_t}{k} = \frac{\widetilde{CF}_t}{20\%} = 5 \times \widetilde{CF}_t .
\end{aligned}
$$

Cost of capital are conditional expected returns (definition).

DCF assumes that these returns are deterministic (assumption).

This allows us to establish a first valuation equation.