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Stochastic Discounted Cash Flow

A Theory of the Valuation of Firms

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Preface to the Second Edition

We were delighted when Springer invited us to prepare a new edition of our book. Several considerations convinced us to take on the task once again:

1. We continue to believe that our work fills a gap in the literature. A glance at the corporate valuation literature reveals a clear bifurcation.

   – Papers aimed primarily at practitioners tend to focus on the notion of the cost of capital. Uncertainty is often reduced to an expected value, whereas detailed descriptions of the distribution of risky cash flows are rarely provided.
   – By contrast, more theoretically oriented papers avoid the term "cost of capital" altogether—e.g., in Cochrane's well-known book *Asset Pricing*, the term appears only once, and Cochrane describes its application as "difficult to use correctly for multiperiod problems." Theorists focus on the so-called pricing kernel (which is economically equivalent to risk-neutral probabilities). This approach, however, is not without problems: The cost of capital is directly observable, whereas the pricing kernel is not and emphasizing the pricing kernel introduces additional challenges for empirical work.

   With our book, we connect the concept of the cost of capital with that of risk-neutral probabilities. Our approach provides a detailed description of the underlying uncertainty and accommodates this synthesis without difficulty. The idea itself is not new—it has been known for more than half a century from option-pricing theory. Against this background, our goal is to restore the concept of the cost of capital to its rightful place in financial analysis. Yet we must acknowledge that we still stand rather alone in pursuing this idea.
2. In recent years, we have continued to work on this subject and are pleased to incorporate the insights we have gained. This keeps the content up to date. In particular, this applies to our reflections on stochastic costs of capital—which we have not encountered elsewhere in the literature—as well as to several other ideas. We also owe much to the many students whose critical questions have helped us explain difficult topics with increasing clarity.
3. Finally, in this edition we have modified certain assumptions, as we believe they tended to create unnecessary confusion. In earlier versions of the book, we defined the

growth rate of cash flows in the most general way possible, allowing it to vary at every point in time (denoted by g_t). While such generality may be scientifically interesting, it is pedagogically unwieldy. Moreover, it complicates empirical work, since one must first specify the functional form of the time dependence before analyzing data. We have therefore decided to abandon this assumption.

The same applies to several terminological choices. In earlier editions, we referred to “(weak) autoregression” of cash flows. Yet autoregressive random variables differ in important mathematical respects from what we intend to describe. It is therefore more accurate to speak of a martingale property of the cash flows. We have also revised terminology in the chapter on insolvency. For example, bankruptcy denotes the legal process of insolvency, which is not our focus here. We have documented these adjustments relative to earlier editions in the text.

The world has changed considerably since we first published our manuscript with Wiley. Today, students in classes no longer hold books in their hands; instead, they read on tablets and write with digital pens. We also believe in the future of open-access formats, although we admittedly regret the loss of royalties from earlier books (which, however, were never substantial). Springer provides the framework for a digital edition as well as the open-access format.

We remain true to ourselves in not expanding the book—despite current fashion—to empirical questions or to other theories sometimes associated with discounted cash flow. There is, however, one notable exception, related to our reflections on stochastic costs of capital. We consider it worthwhile to discuss “excess volatility” in some detail, although we omit the calculations it entails.

Writing a book together is never quite as efficient as we imagined at the beginning. Long stretches of agreement are broken up by debates over words, statements, and entire sections, and since English is not our first language, we began with a generative, pre-trained habit of expression—as if guided by a robot. Over time, that conversation transformed into a draft we could live with. If some passages still sound like a dialogue (shaped by our chats) that simply reflects how this text came to life. In that process, we have been lucky to rely on many hard-working students and their questions, which helped our presentation evolve and improve.

Finally, the authors thank Freie Universität Berlin for its support, which made this revision possible. Andreas Löffler is also grateful to Stefan Steins for many insightful discussions that significantly improved, in particular, the section on insolvency in this book.

Berlin, *Lutz Kruschwitz*
July 2026 *Andreas Löffler*

Preface

We are very pleased that Springer was ready to publish a revised version thirteen years after the our book was published by Wiley. The following items have been changed:

– The way we dealt with the problem of transversality in the Wiley edition never really got us satisfied. We now believe that we have found a way to deal with this issue much more convincingly than before.
– The same applies to sections where we discuss companies that can go bankrupt. We have incorporated new findings where they seemed important to us.
– We have extended the problems and included the sample solutions in the text; a separately available Solution Manual no longer appeared up to date to us.
– The book is now open access. Such a feature did not exist a decade ago and we are extremely pleased that Springer makes possible this form of publication today.
– Last but not least, we would like to draw the reader's attention to the fact that the title of the book has changed—seemingly only slightly. For us authors, however, this is an important matter. We have found that the words "Discounted Cash Flow" are too reminiscent of the literature aimed at practitioners whose job it is to find simple ad hoc solutions to the many problems of company valuation. We are academics and see our mission in finding logically stringent answers to complicated questions. This is sometimes quite far away from what practitioners have to cope with. Unlike practitioners, we cannot accept "sticking tax rates to capital costs in any arbitrary way" or ignore the probabilistic basis of risk and uncertainty. Rather, we want to concentrate on the scientific foundations of calculation under uncertainty and focus on the stochastic questions of valuation. Unfortunately, it then follows that we will inevitably have to ignore some practically important details in our analysis. But then the title should not fake what the book does not deliver. Therefore, from now on the book is called "Stochastic Discounted Cash Flow."

Undoubtedly, empirical questions of company valuation are currently in vogue. However, we sincerely hope that there will be a return to analytical questions in the not too distant future.

Berlin, *Lutz Kruschwitz*

November 2019 *Andreas Löffler*

Preface to the Wiley Edition

We started the work on the subject in 2002. A first draft of the manuscript was published as discussion paper of the University of Hannover. Scott Budzynski translated this first draft and it was subsequently used for several courses in Hannover and as well as in Berlin. We thank our students who carefully read the manuscript and helped us to eliminate a lot of mistakes. Further thanks go out to Dominica Canefield, Erik Eschen, Inka Gläser, Matthias Häußler, Anthony F. Herbst, Jörg Laitenberger, Petra Pade from GlobeGround, Marc Steffen Rapp, Stephan Rosarius, Christian Schiel, Andreas Scholze, Saskia Wagner, Martin Wallmeier, Stefan Weckbach, Jörg Wiese, Elmar Wolfstetter, and Eckart Zwicker for helpful comments. Arnd Lodowicks and Thorsten Rogall contributed two financial policies in the corporate tax chapter.

We would like to thank the *Verein zur Förderung der Zusammenarbeit von Lehre und Praxis am Finanzplatz Hannover e.V.* Their generous support made our work much easier. We thank our wives Ingrid Kruschwitz and Constanze Stein, who never complained when we sat up the whole night in front of our computers.

Berlin and Hannover, February 2005	*Lutz Kruschwitz* *Andreas Löffler*

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α	share transferred upon insolvency
$\widetilde{A}$	amount of retained earnings
$\widetilde{Accr}$	accruals
c	coupon rate
$\widetilde{D}$	market value of debt
$\underline{\widetilde{D}}$	book value of debt
Div	dividend
$\underline{\widetilde{e}}$	increase in subscribed capital
$\widetilde{E}$	market value of equity
$\underline{\widetilde{E}}$	book value of equity
$\mathrm{E}[\cdot]$	expectation
$\mathrm{E}_t[\cdot]$	(conditional) expectation at t
$\widetilde{EBIT}$	earnings before interest and taxes
$\widetilde{CF}$	(free) cash flow
$\widetilde{CF}^{l}$	(free) cash flow of levered firm
$\widetilde{CF}^{u}$	(free) cash flow of unlevered firm
$\widetilde{GCF}$	gross cash flow before tax
$\widetilde{I}$	interest payment
$\widetilde{Inv}$	investments
k	cost of capital of untaxed firm
$\widetilde{k}^{D}$	cost of debt
$k^{E,l}$	cost of equity of levered firm
$k^{E,u}$	cost of equity of unlevered firm
$\widetilde{k}^{\varnothing}$	WACC, type 1
$\widetilde{l}$	leverage ratio to market values
$\underline{\widetilde{l}}$	leverage ratio to book values
$\widetilde{L}$	debt–equity ratio to market values
$\underline{\widetilde{L}}$	debt–equity ratio to book values
$\widetilde{Pr}$	debt repayment (principal)
r_f	risk-free interest rate
τ	tax rate
$\widetilde{Tax}$	tax payment
$\widetilde{Tax}^{l}$	tax payment of levered firm
$\widetilde{Tax}^{u}$	tax payment of unlevered firm
$\widetilde{V}^{l}$	market value of levered firm
$\widetilde{V}^{u}$	market value of unlevered firm
$\underline{\widetilde{V}}$	book value of unlevered firm
$\widetilde{WACC}$	WACC, type 2

List of Definitions, Theorems etc.

Chapter 1
Basic Elements: Cash Flow, Tax, Expectation, Value, Cost of Capital

"Eine neue wissenschaftliche Wahrheit pflegt sich nicht in der Weise durchzusetzen, daß ihre Gegner überzeugt werden und sich als belehrt erklären, sondern vielmehr dadurch, daß die Gegner allmählich aussterben und daß die heranwachsende Generation von vornherein mit der Wahrheit vertraut gemacht ist."
Max Planck, Autobiography

"Science progresses funeral by funeral."
in Paul Samuelson's pithy phrasing.

Abstract In this chapter, we describe precisely and carefully all model properties, assumptions, and definitions required for DCF.
We discuss definitions of cash flow, describe how a stylized tax system should be modeled, and address uncertainty using five rules for conditional expectations. The firm's cost of capital is defined as a conditional expected return, from which we derive a first valuation equation for pricing cash flows over time. We then introduce the concept of no-arbitrage and its link to the Fundamental Theorem of Asset Pricing, and finally examine the infinite-horizon case, where a transversality condition rules out bubble solutions.

1.1 Introduction: A Stochastic Approach to Discounted Cash Flow

The valuation of firms is an exciting topic. It is interesting for economists engaged in either practice or theory, particularly for those in finance. Amongst practitioners, it is investment bankers and public accountants, who are regularly confronted with the question of how a firm is to be valued. Rappaport's discussion of shareholder value highlighted that traditional accounting numbers offer little insight into whether management has actually created or destroyed value. Instead, it is the firm's market value that provides a meaningful measure of managerial performance. This pragmatic view serves practitioners well, but academic interest in valuation stems from different considerations.

On closer examination of how finance theorists assess firm value, one finds that they do not primarily regard firms as institutions that combine production factors to produce goods or services. The economic activities themselves are not examined in further detail. Rather, the focus lies on the income available to financiers while the firm's role in satisfying consumer needs is of merely secondary relevance. What is decisive is the amount of payments and its distribution among the owners and creditors. The income a company generates for its owners determines its value. In the end, a firm is nothing more than a risky asset, or a portfolio of assets. The valuation of firms deals with nothing else than the question as to what economic value future earnings have today. In short: more is better, sooner is preferable, and lower risk adds value.

The literature on valuation of firms promotes logical, quantitative methods aimed at determining today's value of future free cash flows. In this respect, the valuation of a firm is identical with the calculation of the discounted cash flow, which is often only given by its abbreviation, DCF. There are, however, different coexistent versions that seem to compete against each other. Entity approach and equity approach are thus differentiated. Acronyms are often used, such as APV (adjusted present value) or WACC (weighted average cost of capital), whereby these two concepts are classified under the entity approach.

We see it as very important to systematically clarify the way in which these different variations of the DCF concept are related. Why are there several procedures and not just one? Do they all lead to the same result? If not, what are the economic differences? If so, for what purpose are different methods needed? And further: do the known procedures suffice? Or are there situations where none of the concepts developed up to now delivers the correct value of the firm? If so, how is the appropriate valuation formula to be found? These questions are not just interesting for theorists; even the practitioner who is confronted with the task of marketing her results has to deal with them.

When finance theorists discuss the valuation of risky assets, various conceptual complications arise that make it difficult to apply their results directly to practical firm valuation. Theoretical economists usually concentrate on specific details of their object of examination and leave out everything else which they consider to be less important at the moment. There have been models in which—for purposes of simplification—it is supposed that the firm to be valued will survive for exactly one year. Or you find models in which it is required for convenience's sake that the firm goes on forever. But in return for that, it yields cash flows which remain the same and it does not have to pay taxes. In yet other models, it is assumed that although a very simple tax is brought to bear on the business level, the shareholders are, however, spared any taxation. It is supposed in further models that the price of an asset follows a stochastic process, which the analyst can describe very accurately and over which she is methodically (mathematically) in total control. Such simplifications and specializations are part and parcel of theoretical work. Those approaches are not only widely used but also highly advantageous. Yet they are not always appropriate for practical applications in firm valuation. This is why considerable effort must be made to move beyond the purely theoretical framework, which rests on highly simplifying assumptions. Instead, the aim should be to develop valuation equations that incorporate simplifications only where essential.

The article by Modigliani and Miller represents, for instance, an important starting point for traditional DCF theory. Two things are characteristic for this contribution: first, a very simple corporate income tax is depicted; second, the leverage ratio of the firm is measured in market values. If you now have to value a German firm, for example, the results of the Modigliani-Miller model cannot simply be applied. Instead, you have to carry out appropriate adjustments. First, the German system of corporate taxation is somewhat more complicated. And second, it could be that the managers of the firm have decided in favor of a financing policy in which the leverage ratio is measured on balance-sheet basis. The question must then be asked, “how are the formulas to be changed?” The theoretical literature so far offers no clear path to follow.

If the theory does not provide an answer, practitioners are left with no other choice than to ad hoc adjust the valuation equations according to their judgment so that they do justice to the present situation as far as they are convinced. For the practitioner, the theory can only be a guideline to go by anyway. They are used to taking matters into their own hands in order to make the theory at all useable.

The theorists are not under the same time constraints as the practitioners. They are obligated to the truth, not their mandates. This is the reason it is not allowed for them to simply change valuation equations ad hoc, which were developed under the specific circumstances of a model, if the original conditions no longer prevail. They must instead abide by rules that ensure their assertions are correct. As a first step, the new conditions are to be described in an orderly way. Building on this foundation and other internally consistent assumptions, the theorist seeks to derive the valuation equation for the case at hand using logical and mathematical reasoning. It is only in observing these conventions that they have the right to recommend a specific valuation equation as appropriate. And it is only in observing these rules that a third party has the possibility to check whether a valuation procedure in fact deserves the predicate “suitable.” We are convinced that whoever does otherwise risks being accused of having no scientific ground to stand on.

We will attempt in this book to stick to the line of thought just described. We will thus not ad hoc draw up valuation equations, but rather call to mind the theoretical groundwork upon which these were gotten. We see no other serious alternative in this matter. Readers, who take the trouble to follow along with us will indeed be rewarded with a lot of discoveries, which are at once formally precise and also economically interesting.

In closing, we want to get a little more specific about the “long way round” we have propagated and impart what we regard as particularly characteristic of our methodology. More than anything else there are four points, which differentiate our depiction of the DCF concept from that of the literature up until now:

No arbitrage Certain paradigms dominate finance theory today. These include, for instance, expected utility theory, the concept of perfect markets, the postulate that markets are free of arbitrage, and the equilibrium concept, to name just a few. No empirically minded theorist would claim that any of these are empirically representative. We are well aware that managers and investors do not always behave rationally. This observation forms the basis for an important field of research that is now known as behavioral finance.

The assumption of homogeneous expectations among investors is a defining feature of the perfect-capital market framework. Yet it is evident that, in reality, market participants operate under asymmetric information. Principal–agent models, which explicitly take this into account, have therefore gained considerable importance in finance theory over the past decades.

As far as the development of valuation equations is concerned, finance theorists have been successful only when they have strictly followed the principles of the neoclassical paradigm. This paradigm rests without ifs, ands or buts on the assumption that there is no free lunch in the market. Although arbitrageurs exist in reality, theorists have almost universally refrained from incorporating such opportunities into their models. Allowing them would render any discussion of prices meaningless, since prices could no longer serve as the basis of exchange if everyone could, in effect, "print their own money" at will. The no-free-lunch principle therefore constitutes the indispensable cornerstone of the neoclassical framework. For this reason, all valuation equations in this book are derived under this condition. We are convinced that this principle has not always been consistently observed in addressing the valuation of firms, and we shall make this explicit where relevant.

Cost of capital Cost of capital is definitely one of the key concepts in finance. Surprisingly, there is often no definition of this term in the literature that is precise enough to be used with logical operations to derive valuation equations, particularly in a multiperiod context. Since we regard cost of capital as a central construct, we have chosen to begin our considerations with its clarification.

Consequently, several statements that are regarded in the literature as obviously true have to be proven by us.

In fact, in modern theoretical finance the concept of cost of capital is far from uncontested. Its usefulness is often questioned, while in applied contexts it is usually taken for granted. We shall return to these conceptual difficulties in the following sections.

Data Given As far as we can see, the information available to the analyst about the firm to be valued plays no systematic role in the DCF literature. Yet it is precisely this information that determines how the analyst should perform the calculation. Therefore, whenever we develop a valuation equation, we will specify in detail which information is assumed to be available.

Uncertainty of Cash Flows We have found that in the process of evaluating a company, many practitioners attach little or no importance to the stochastic structure of future cash flows. Practitioners are much more likely to limit themselves to estimating the expectations of cash flows. Taking this implication seriously would rule out valuation concepts explicitly based on the stochastic structure of payment patterns.

We thus concentrate on results where the stochastic structure recedes into the background, and the expectation of cash flows emerges as the key element. That will play an important role in our analysis.

The first version of our book has been published some 20 years ago. Since then we have worked continuously on the subject and added new insights to the second edition. We removed mistakes and typos, added a new chapter on transversality and were able to enrich our findings on insolvency. New literature was also included. Some of our ideas have since then found its way into textbooks, although we still think that a systematic and concise approach as we prefer it is still missing.

Following earlier convention, we marked all random variables with a tilde, i.e., $\widetilde{X}$. Today, such a notation clearly looks old-fashioned, typographically complex and in the modern literature the use of the tilde is discouraged. Notwithstanding this fact, we decided to mark random variables with a tilde and this mainly for didactic reasons. Otherwise, it may take the inexperienced reader some time to see when an equation like

$$\mathrm{E}[XY] = X\,\mathrm{E}[Y]$$

is valid and when it is not.

There has been—certainly accelerated by the COVID-19 pandemic—a strong increase in demand for supplementary materials such as videos, and the authors of this book have always enjoyed experimenting with new media (one of the authors recorded his first lecture as early as 2005). We provide extensive additional material for those who wish to use this book for teaching or self-study. Slides and further resources are available on our website:

www.wacc.info

1.2 Fundamental Terms

Valuation is being talked about everywhere. Finance experts, CPAs, investment bankers, and business consultants have been debating the pros and cons of discounted cash flow (DCF) methods for years. This book joins that debate by contributing a theoretical perspective.

Anyone engaging with the DCF approach will inevitably encounter a few recurring terms. It is typically said that the valuation of a firm involves the discounting

– of its future payment surpluses
– after consideration of taxes
– using the appropriate cost of capital.

Three things need to be clarified: first, what exactly constitutes a cash surplus; second, how taxes are treated; and third, how the cost of capital is defined.

The payment surpluses are also called cash flows. Nowhere in the literature is this term clearly defined. So it is safe to say that no two economists mean exactly the same thing when they talk about cash flows. The reader of this book might expect that we will go into elaborate detail on how cash flows are determined. We must disappoint such expectations. Essentially, we will be limiting ourselves to working out the difference between gross cash flows and free cash flows.

It is relatively clear what is meant by taxes in the context of business valuation. The lawmakers leave no doubt as to which payments are due to them. Furthermore, it is known to all those involved in firm valuation that taxes on profit are to be taken into particular consideration. Finally, every analyst knows that there are taxes on the corporate as well as on the private level of business. In Germany, for instance, one must think about corporate and trade tax on the corporate level, and about income tax on the level of the financiers. This book is not, however, intended for readers who are interested in the details of a particular national tax system. Therefore, we do not plan to individually present the British, German or US American tax systems. On the contrary, we will base our considerations on a stylized tax system. Some readers may have different expectations regarding this point as well.

As a rule it remains rather unclear in business valuation what is meant by cost of capital. Even those consulting the relevant literature will, in our view, find no precise definition of the term. This brings us head-on with the question as to what cost of capital is.

Every theory of business valuation rests on a model. Any such model comes with specific structural features, which we describe in this section. In the following sections, we discuss cash flows, taxes, and the cost of capital in more detail.

1.2.1 Cash Flows

To apply the DCF approach, the analyst must estimate the company's future cash flows. This immediately raises two distinct questions: first, what exactly is meant by cash flow; and second, how can those future cash flows be forecast? The former is a matter of definition, whereas the latter concerns prognosis. In this section, we focus on the first issue.

Gross Cash Flow Gross cash flows refer to the net inflows generated through normal business operations. These may either be distributed to financiers or retained within the firm and thus reinvested. When referring to the financiers, we will later distinguish between shareholders and debt holders, with payments taking the form of interest and debt service on the one hand and dividends and capital repayments on the other. If taxes have not yet been deducted from the gross cash flow, we speak of gross cash flow before taxes, see Table 1.1.

Table 1.1 From EBIT to gross and free cash flow.

	Earnings before interest and taxes (EBIT)
+	Accruals
=	Gross cash flow before taxes
−	Corporate income taxes
−	Investment expenses
=	Free cash flow
−	Interest and debt service
−	Dividend and capital reduction
=	Zero

To determine the gross cash flow for a past accounting year, one usually relies on the firm's financial statements. One studies balance sheets, income statements and, most likely, cash-flow statements as well. The way in which one needs to deal with these individually depends heavily upon which legal provisions were used to draw up the annual reports, and on how the existing law structures in place were used by the managers of the firm. It makes a big difference if we are looking at a German corporation, which submits a financial statement according to the total cost format, and in doing so follows the IFRS (International Financial Reporting Standards), or if it concerns a

US American company, which reports according to the cost of sales format and pays heed to the US-GAAP (Generally Accepted Accounting Principles). There is no uniform procedure for calculating gross cash flow across firms. Thus, we have justified why we do not elaborate further on the determination of gross cash flows.

Free Cash Flow Companies must continually invest if they want to stay competitive. These investments are usually subdivided into expansion and replacement investments. Expansion investments ensure the increase of capacities and are indispensable if the firm should grow. The replacement investments, in contrast, ensure the furtherance of the status quo. They are, hence, usually based on the accruals. We assume that the firm being valued invests in every period. Sensibly enough, those are only investment projects that are attractive from an economic perspective.

We refer to the difference between gross cash flow after taxes and the amount of investment as the firm's free cash flow. This amount can be paid out to the firm's financiers, namely the shareholders and the creditors.

Projection of Cash Flows In practice, the analyst must devote considerable time to forecasting future cash flows: we already mentioned that it is not the historical payment surpluses that matter to the firm being valued, but rather the cash flows that it will yield in the future. The work of finance theorists is generally of limited use for this important activity. We will not discuss practical questions related to the prognosis of cash flows in this book at all.

1.2.2 Taxes

Income-, Value-Based-, and Sales Tax A company is subject to several types of taxes. Economists typically distinguish among income taxes (for example, personal income tax), value-based taxes (for example, real-estate tax) and sales taxes (for example, value-added tax and numerous others). For the purposes of this book, sales taxes play no significant role—they simply depict a component of the cash flow and are otherwise of no further interest. Of course, if such taxes cannot be fully passed on to consumers, they may affect prices and quantities and thus the firm's operating cash flows. We abstract from these incidence effects here. Value-based taxes are also not usually discussed in-depth in the literature on valuation of firms. We follow this convention and are concentrating largely just on income taxes.

Corporate and Personal Income Taxes Income tax is imposed on the firm level, as well as upon the shareholder level. In the first case we will speak about a corporate income tax, and in the second about personal income tax. In the United States, a corporate income tax is to be thought of with business tax; in addition, income tax accrues on the shareholder level (at the federal and local levels).

Our readers should not expect that we will go into detail on either the US American or other national tax systems. In this book we do not intend to treat the particularities of national tax laws with their immeasurable details. We have two reasons for this. For one, national tax laws are subject to constant changes. Every such change would require a new edition of the book. We are concerned here with a general theory, which is able to deal with the principle characteristics of tax law. Secondly, we would thus not only have to deal with the integration of one single tax law into the DCF approach, but rather of the tax laws of every important industrial nation around the world. This would, however, overstep the intended breadth of the book.

Anyone who equates a firm's value with its marginal price from an average investors's perspective has no alternative than to consider the corporate as well as the personal income taxes.[1]

Features of a Tax To identify a tax more precisely, three characteristics must be kept in mind—features that can be observed in every type of tax. The first question, therefore, is who must pay the tax; this is referred to as the tax subject. The tax base expresses how the object of taxation is quantified. And finally, the tax scale describes the functional relation between the tax due and the tax base. We are characterizing the corporate income tax and the personal income tax as well in the following with regard to the three named characteristics.

Tax Subject The tax subject describes who has to pay the tax. In the case of the corporate income tax it is the firm, which is to be valued, while the object of taxation are the business activities of the firm. In the case of the personal income tax it is the financiers (owner as well as debt holder) who are the tax subject. The object of taxation is the income generated by the firm or by other activities, particularly those on the capital market.

Tax Base The corporate income tax is calculated according to an amount that is commonly referred to as profit. If one thinks of the US corporate income tax, one effectively envisages the tax profit. Regarding the personal income tax the gross income minus some expenses form the tax base.

Tax Scale Once the tax scale is applied to the tax base, the resulting amount determines the tax due. Proportional and non-proportional scale functions are commonly observed. In what follows, we will work with a proportional (flat) tax scale and disregard both allowances and exemption thresholds. The tax due is ascertained by multiplying the tax base by a tax rate, which we assume is independent of the tax base.

This assumption is, of course, unrealistic. We are not aware of any income tax system worldwide in which the tax scale is truly proportional. Even the smallest allowance renders the tax function non-proportional. Although a few studies on nonproportional taxes have appeared in recent years, they remain rare and can almost be counted on one hand. Hence, we will uphold a flat tax.

In the meantime, at least in the deterministic case, it has been clarified how to handle non-proportional taxes. It turns out that the considerations for a linear tax can largely be transferred to the non-proportional case if a few plausible assumptions are met (for example, that the tax function is convex). In Sect. 3.1.5, we have provided a more detailed discussion of this topic.

[1] In Germany that corresponds to the viewpoint of the profession of chartered accountants, see Institut der Wirtschaftsprüfer in Deutschland (2008).

What is more, we will assume that the tax rate at the time of valuation is known and absolutely unchangeable! That is a far-reaching assumption, and we are entirely aware of the resulting limitations. We are not aware of any systematic discussion of stochastic tax rates in the literature, apart from isolated cases, and none at all within DCF approaches. There is a gulf here between theory and practice that we will not be able to bridge over in this book. The practice-oriented reader may find our approach rather remote from real-world conditions. In this book, we will later frequently refer to deterministic and stochastic tax advantages. If we state this here, and then later follow through on it, the manager may suspect (not unjustifiably) the most important source of uncertainty to be future unknown tax rates. Regardless, all known DCF approaches rule out just this source of uncertainty before we even begin. We still have a wide field of research ahead of us.

1.2.3 Cost of Capital

We do not know if this book's reader is particularly interested in a precise definition of cost of capital. We are convinced that it is of considerable importance for the discussion of DCF methods among theorists. We would be happy, if our readers were sympathetic to this perspective, or at least came to understand it while reading the book.

Cost of Capital as Returns In order to make our considerations more understandable, let us leave out uncertainty. The company that is to be valued may promise deterministic cash flows for the future, which we denote by $CF_1, CF_2, \ldots$. We will gain a preliminary understanding of the notion of cost of capital by examining the role it should play. Cost of capital serve for the determination of the company's value. For this purpose the fixed cash flows are discounted with the (probably time-dependent) cost of capital. A valuation equation would look, for example, like the following:[2]

$$V_0 = \frac{CF_1}{1+k_0} + \frac{CF_2}{(1+k_0)(1+k_1)} + \ldots , \tag{1.1}$$

where $k_0, k_1, \ldots$ are the cost of capital of the zeroth, first, and all further periods and V_0 is the company's value at time $t = 0$. In the course of our book, we will see that we repeatedly need an equation for the future value of firm V_t at $t > 0$. It would be convenient if Eq. (1.1) could be used analogously at later times. For that we had

$$V_t = \frac{CF_{t+1}}{1+k_t} + \frac{CF_{t+2}}{(1+k_t)(1+k_{t+1})} + \ldots . \tag{1.2}$$

And one obviously gains an understanding of the structural relationship by which such future values are to be determined.

Eq. (1.2) can be used to infer the relation

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[2] We are, for the moment, disregarding questions related to an infinite time horizon. These will be discussed in detail in Sect. 1.4.5.

$$k_t =_{\text{Def}} \frac{CF_{t+1} + V_{t+1}}{V_t} - 1 , \tag{1.3}$$

that gives us a basis for a precise definition of the cost of capital as future *return*. The economic intuition of such a definition is most easily revealed if one imagines that an investor at time t acquires an asset for the price V_t. At time $t+1$, this asset may yield a cash flow (a dividend) of CF_{t+1} and immediately afterwards be sold again for the price of V_{t+1}. The return of such an action is then precisely given through the Eq. (1.3).

Obviously, the definition of the cost of capital (1.3) and the application of the valuation Eq. (1.2) for all times $t = 0, 1, \ldots$ are two statements, which are logically equivalent to each other. If it is decided to understand cost of capital as return in the sense of Eq. (1.3), then the valuation statement (1.2) is straightforward. The inversion is true as well: starting out from the valuation statement (1.2), it is implied that the suitable cost of capital is indeed a return. This simple idea will be the red thread throughout our presentation.

Cost of Capital as Yields Up to this point, we have considered the firm's cash flows in their entirety. To sharpen our understanding, however, let us now shift perspective and focus on a single future cash flow. By first valuing this isolated payment, we prepare the ground for subsequently aggregating all such cash flows into the firm's overall valuation.

Let us suppose for a moment that it is possible at time t to pay a price $P_{t,s}$ and in return to earn nothing else than a dividend in amount of CF_s (in which $s > t$). If under $\frac{1}{(1+\kappa_{t,s})^{s-t}}$ the price of a monetary unit is understood that is to be paid at time t and which is due at time s, then we designate $\kappa_{t,s}$ as *yield*. The following relations are valid for such yields,

$$\begin{aligned} P_{t,t+1} &= \frac{CF_{t+1}}{1+\kappa_{t,t+1}} \\ P_{t,t+2} &= \frac{CF_{t+2}}{(1+\kappa_{t,t+2})^2} \\ &\vdots \end{aligned}$$

The value of a firm at time t, which promises dividends at times $s = t+1, \ldots$ could be written in the form

$$V_t = \frac{CF_{t+1}}{1+\kappa_{t,t+1}} + \frac{CF_{t+2}}{(1+\kappa_{t,t+2})^2} + \ldots . \tag{1.4}$$

But to the contrary of the valuation Eq. (1.2), the formal structure of (1.4) cannot be used at time $t+1$. The yields of time t will certainly be different from the yields one period later, i.e.,

$$V_{t+1} = \frac{CF_{t+2}}{1+\underbrace{\kappa_{t+1,t+2}}_{\stackrel{?}{=}\kappa_{t,t+1}}} + \frac{CF_{t+3}}{(1+\underbrace{\kappa_{t+1,t+3}}_{\stackrel{?}{=}\kappa_{t,t+2}})^2} + \ldots .$$

For our purpose it is only appropriate to understand cost of capital as returns and not as yields.

Those who are used to work with empirical data will not hesitate to agree with our definition of the term. At any rate, in all empirical examinations known to us, returns are always determined when cost of capital is to be calculated. It is much more difficult to estimate yields or discount rates (to say nothing of risk-neutral probabilities which will be introduced shortly). Therefore, defining cost of capital as returns is suitable.

The question must now be asked how the concept of cost of capital can be defined under uncertainty. An answer must wait until a later section.

1.2.4 Time

In order to illustrate a basic idea of the DCF model, let us use an agricultural analogy: a cow is worth as much milk as it gives. For businesses and their market value, that means a firm's market value is determined by its future payment surpluses. If this is accepted, then the question arises as to how long a firm stays alive.

Lifespan So long as nothing to the contrary is known, one cannot go wrong in assuming that the firm will be around for more than a year. Such a vague supposition is of little help. As a rule it can be said that firms are set up for the long-run, and that most investors involved in the purchase of companies—and who fall back on procedures of business valuation for this means—have an investment horizon, which clearly overstretches one year. However, we are running in circles, because whether we assume the business will survive for more than a year, or think that it will remain active until "however long," the picture is still pretty unclear.

Coming to a head, this raises the question of whether business valuation should assume that firms have a finite horizon or operate in perpetuity. At first glance, the notion of an infinitely lived firm may seem peculiar.

Regardless, there are worthwhile arguments in favor of the fiction of the perpetual company. If one values a firm, which only has a limited life-span, its end-date must be determined. An exact answer (apart from very few exceptions) would be impossible. Moreover, on the last day of the firm's history, a residual value will be paid to the owners. If you wanted to answer the question as to how the firm's residual value is to be determined, you would have to fall back upon the subsequent payments to be attained—that certainly does not fit into our assumption anymore that the world is just about to end. That is why an exact calculation of the residual value would likewise not succeed. If it can be shown that it makes no difference worth mentioning whether you operate under the premises of a business life-span of, let us say, 30 years, or that the business is incessantly active, then the fiction of a perpetually active firm, albeit objectively false, can be justified for the sake of convenience.[3]

What we want to convey to our readers is that we are working on the basis of an investment horizon spanning many periods, without yet committing ourselves to either a

[3] If cost of capital in the amount of 10% and constant cash flows are implied, then the first 30 years explain virtually 95% of the firm's total value.

finite (denoted by T) or an infinite horizon. Since, in practice, if our theory were to be implemented, we would propose an infinite planning horizon.

Trade- and Payment Dates It is necessary to specify the trade- and payment dates of our model. An investor, who owns a share receives the cash flow at the instant immediately preceding t (see Fig. 1.1). If she sells this security at t, then the buyer always pays a "price ex cash flow." Although this arrangement is commonly used within the framework of the DCF approach, we explicitly stress it here. It leads to not being able to illustrate particular trading strategies in our model. For dividend stripping, for instance, a share would have to be bought immediately before the dividend payment—a comportment our model does not allow.

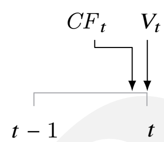

Fig. 1.1 Prices are always ex cash flow.

Continuous- or Discrete-Time Model We have chosen to analyze the firm over a time frame of several periods. But should our model now be continuous-time or discrete-time?

In order to make the difference between both types of models clear, let us look at a firm with a finite life-span of T years. If we use a time scale, in which there are only times $t = 0$ (today), $t = 1$ (one year from today), . . . , $t = T$ (T years from today), then the model's framework is discrete. We could, without question, divide up each year into quarters, months, weeks or even days. In the last case, we would run the time index t from 0 to $365\,T$, and since the days are countable, we would still be dealing with a discrete-time model. The more minutely we choose to divide up the time, that many more subperiods there are in a year. But only after we let the number of annual subperiods grow beyond all limits, so that the number of time intervals can no longer be counted, would we be looking at a continuous-time model.

After we have gotten a good enough idea as to where the difference between discrete-time and continuous-time models lies, we turn back again to the question as to which type of model we should choose. In so doing, we find out that we do not have any criteria by which we can ascertain the advantages and disadvantages of one type or another.

One could get the idea that this continuous modeling is unpractical, since just as the cow cannot be continually milked, neither can a firm pay out dividends incessantly. So, let us instead say that the cow is milked once a day, and a corporation pays dividends once a year, for instance. Such intermittent events can, however, be included within a continuous-time model, as well as a discrete-time model. We must think of something else.

In the modern finance literature, continuous-time models have experienced a notable boom. They are much more popular than discrete-time models. There is quite a compelling reason for this. It is less a matter of whether continuous-time models are more realistic—as the question of the realism of economic models has been clearly addressed since

Milton Friedman: the core purpose of economic models is that they are not realistic. Therefore, this should not guide our choice between the two models. Both discrete-time and continuous-time models are inherently unrealistic. These considerations do not provide us with further clarity.

Continuous-time models are more popular because certain calculations (for example, option pricing) can be performed more easily with them. The known discrete-time models in the literature require more cumbersome methods for these calculations.[4] This criticism does not seem to convince us when it comes to company valuation. We have tried to perform the proofs and calculations solely by using five clearly defined computational "rules"—that is, streamlined calculation techniques for working with discrete-time models (see Sect. 1.3.2)—to keep the effort minimal.

Additionally, the mathematical tools required in continuous-time models are far more demanding compared to those used in discrete-time models. We can assert this confidently because we have formed our own impression: our book Löffler and Kruschwitz (2019) serves precisely the purpose of clarifying, from an economist's perspective, what is actually behind the assumptions of a Brownian motion. We believe that for corporate valuation, this elaborate apparatus of continuous-time calculus is not necessary. Instead, all that is relevant for valuation can also be discussed within a simple discrete-time model without Brownian motion. For didactic reasons, we therefore believe that discrete-time models are better suited for our purposes. Consequently, only these models are presented here.

There is more that, in our view, speaks against stochastic differential equations and in favor of discrete-time models. Those who use Brownian motions are essentially employing nothing more than an (infinitely refined) binomial tree.[5] These binomial trees have a very simple structure: uncertainty arises at each node, but in its simplest conceivable form. Whereas uncertainty in general may involve many possible outcomes, the binomial tree restricts this number to exactly two—no more than absolutely necessary. That this is an economically very simple structure is not apparent due to the fog of the mathematically complex apparatus.

Distinguishing the current time $t = 0$ (present) from the future $t = 1, 2 \ldots, T$, the ending time T as seen from today's perspective can or cannot be infinitely far off. The length of a time interval is not dependent upon the situation in which our model will be used. Intervals of one year are typically the case.

Point-in-Time Principle In corporate valuation, it is common to distinguish between the present ($t = 0$) and the future ($t > 0$). It is important to highlight a fundamental principle that applies to all multi-period models: *All considerations take place in the present* (at time $t = 0$). This requirement is also known as the point-in-time principle, sometimes called reference date principle.

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[4] This is very clearly stated, for example, in Cochrane (2005, p. 25f.): "It is often convenient to express asset pricing ideas in the language of continuous-time stochastic differential equations rather than discrete-time stochastic difference equations. . . Even if you want to end up with a discrete-time representation, manipulations are often easier in continuous time. For example, relating interest rates and Sharpe ratios to consumption growth in the last section required a clumsy lognormal approximation; you will see the same sort of thing done much more cleanly in this [continuous-time] section."

[5] See, for example, Löffler and Kruschwitz (2019, p. 44).

We do not move away from the present, but we consider what we know today about a near future ($t = 1$) and what we know today about a distant future ($t = 2, \ldots$). However, it is never about how things actually turn out once the near future has arrived. The point-in-time principle emphasizes what we consider possible today; it is not concerned with how our knowledge may actually expand over time.

If the predicted relationships do not materialize in the future, this can have serious consequences. We will illustrate this more precisely with an example in Sect. 1.2: there, at time $t = 0$, cash flows are forecasted for the time $t = 1$ to be either 90 or 110. No other numbers besides these two values are considered possible today. The theory of the DCF then does not address how to handle situations in which, one period later, cash flows are realized at a different level than 90 or 110. The point-in-time principle prohibits this.

One must realize what this means in our example. If, at time $t = 1$, we observe cash flows that are neither 90 nor 110, then all conclusions drawn from our considerations at $t = 0$ must be discarded. Anyone who has calculated a company value will need to go back to pencil and paper and perform new calculations. The point-in-time principle thus consists in making decisions today in the presence of increasing uncertainty about the future. The appropriate mathematical tool for this is the conditional expectation, which we will introduce in a moment.

1.2.5 Problem

Problem 1.1 Let the world end the day after tomorrow, $T = 2$. Assume that $CF_2 = 100$ is deterministic, but $\widetilde{CF}_1$ is stochastic with $\mathrm{E}\left[\widetilde{CF}_1\right] = 100$. The risk-free rate is $r_f = 5\%$. We show that yields and cost of capital cover different economic terms.

a) Assume that the yield for the first cash flow is $\kappa_{0,1} = 10\%$. Evaluate the company and determine the cost of capital k_0.
b) Assume that the cost of capital for the first cash flow is $k_0 = 10\%$. Evaluate the company and determine the yield $\kappa_{0,1}$.

1.3 Conditional Expectation

As analysts valuing a firm, we find ourselves at time $t = 0$, that is, in the present. The valuation is taking place today, and we can be certain that our knowledge of the company will increase over time. Although we cannot, so to speak, move ourselves out of the present, let us still consider what we know today, and what we will know in the future. This requires the understanding of uncertainty and the notion of conditional expectation.

1.3.1 Uncertainty and Information

We begin with the hardly surprising observation that the future is uncertain. How now? For the variables analyzed by the evaluator, this means that as of today it is still not known what those variables will be. It is not known how many liters of milk the cow will produce tomorrow. It cannot exactly be said what the cash flow of the firm to be valued will be in three years. Instead, there exist many possibilities. We also speak of events, which can influence the amount of the cash flows. Uncertainty in the cash flow is indicated by adding a tilde to its symbol,

$$\widetilde{CF}_t .$$

This type of notation is certainly out of fashion: In many papers and almost all textbooks random variables are nowadays not identified by a particular symbol like the tilde. From a typographical point of view this increases readability. But we believe that for someone who must learn the difference between a random value and a deterministic number it is of great help if the random variable has a clear and visible label. There are transformations that are allowed with deterministic numbers and that fail with random variables—without a unique label this can lead to serious errors. Therefore, we will persist on using the tilde.[6]

States of Nature This representation leaves us in the dark as to what the interested variable is dependent upon. In fact, it is so that we foresee various possible states of nature. Such states could be described, for example, through product's market shares, or unemployment quotas or other variables. The cash flow would then depend on the state variables ω, however defined. If we do not refer to the entire random variable but to the cash flow in one particular state we use the notation

$$\widetilde{CF}_t(\omega) .$$

There are theoretical as well as practical reasons for our not making use of this detailed notation in the following, but instead only using the more simple spelling.

So we will make no statements in our theory at all about whether the number of the possible future states is finite or infinite. Rather, we simply leave it open, as to whether the state space is discrete or continuous. The formal techniques for dealing with continuous state spaces are more complicated than the instruments needed for the analysis of a discrete state space. We are trying to avoid wasting energy as much as possible here.

In our experience, every analyst eventually refrains from making statements about future states of the world. One tries in practice to determine the expectations of the established quantities. And future cash flows, or future returns, are approximately

[6] For example, in the equation $\mathrm{E}[XY] = X\,\mathrm{E}[Y]$ mentioned in the introduction, it is not immediately clear to a lay reader whether X is meant to be a random variable or a constant, and under what conditions the equation does or does not hold. The discussion of the rule introduced next, Rule 5, will exemplify this.

estimated in such a way so that they do not need to rely upon the state-contingent quantities. Then why should we not try to avoid these details in our theory?

For both reasons, we will, in what follows, suppress the dependence of random variables on the states of the world and will not further specify the structure of uncertainty—except in a few cases, such as in our examples.

Notation In order to make ourselves understood, we must introduce a few mathematical variables into the discussion. In so doing, we will use the notation common in current finance literature. The reader who is not used to this, may possibly ask at first why we did not try for a less exacting notation. The formal notation that we are now going to introduce, does take some time to get used to, but is in no way so complicated that one should get scared off. It presents a straightforward and very compact notation for the facts, which need to be described in clarity. We therefore ask our readers to make the effort to carefully comprehend our notation. We promise on our part to make every possible effort to present the relations as simply as they are.

Let us concentrate on, for example, the cash flow, which the firm will yield at time $t = 3$, so $\widetilde{CF}_3$. Between $t = 1$ and $t = 2$, some of the uncertainty surrounding the cash flow is resolved. When speaking about the expected cash flow from the third year and wishing to be precise, we must specify the state of information on which we base our analysis. In earlier versions of this book, we referred, in a mathematically precise way, to the information that the analyst assumes today she will possess at time $t = 1$ as $\mathcal{F}_1$, and denoted the corresponding (conditional) expected value as

$$\mathrm{E}\left[\widetilde{CF}_3 \mid \mathcal{F}_1\right] .$$

The calligraphic $\mathcal{F}$ describes a specific subset of all possible states.[7] However, we do not use these subsets anywhere in our book. We have learned from discussions with students that this mathematically correct but cumbersome notation causes more confusion than it helps in understanding what we mean here. Therefore, we have decided to denote the analyst's expectations about the cash flow of time $t = 3$ from today's presumption of the state of information at time $t = 1$ with

$$\mathrm{E}_1\left[\widetilde{CF}_3\right] \equiv \mathrm{E}\left[\widetilde{CF}_3 \mid \mathcal{F}_1\right] .$$

Both terms stand for the same thing.

If the analyst uses her knowledge at time $t = 2$, she will have a more differentiated view of the third year's cash flow. This view is described through the conditional expectation

$$\mathrm{E}_2\left[\widetilde{CF}_3\right] .$$

How is that to be read? The expression describes what the analyst today believes she will know about the cash flow at time $t = 3$ in two years. So you see, what at first glance appears to be somewhat complicated allows for a very compact formulation. One, which

[7] For those who want to know more: these are the states of the world that cannot yet be distinguished at time t because we do not yet have enough information.

verbally can only be painstakingly presented. Mathematically, the expression

$$\mathrm{E}_s\left[\widetilde{CF}_t\right]$$

is the conditional expectation of the random variable $\widetilde{CF}_t$, given the information at time s.

Classical Expectation What differentiates a classical expectation from a conditional one? With the classical expectation, a real number is determined that represents the average amount of a random variable. Here, however, we are dealing with a situation in which more information about this random variable is already available. The information, for example, could relate to whether a new product was successful, or proved to be a flop. Now the analyst must determine the average amount of a random variable (we are thinking of cash flows) according to the scenario (twice in our example) to establish the conditional expectation. The conditional expectation will no longer be a single number, as in the case of the classical expectation. Instead, it depicts a quantity, which itself is dependent upon the uncertain future: according to the market situation (success or flop), two average cash flows are conceivable. Summing up this observation, we must consider the following: the conditional expectation itself *can be a random variable*! This differentiates it from the classical expectation, which always results in a real number.

In our valuation theory, we will often be dealing with conditional expectations. Some readers may therefore be interested in getting a clean definition and being told about important characteristics of this mathematical concept in detail. We will now disappoint these readers, as we do not intend to explain in further detail what conditional expectations are. Much rather, we will just describe how they are to be used to calculate with. There is one simple reason why we deliberately exclude this. We have, until now, avoided describing the structure of the underlying uncertainty in detail. If we now wanted to explain how a conditional expectation is defined, we would also have to show how an expectation is calculated altogether and what probability distributions are. But we do not need these details for business valuation. A few simple rules suffice. The entire mathematical apparatus can be left in the background. Likewise, to be a good driver, you do not need to worry about the physics of electric motors or understand how batteries store electrical energy. It is enough to read the owner's manual, learn the road signs, and get some driving experience. Experts must forgive us here for our crude handling.[8]

1.3.2 Rules

In the following, we will present five simple rules for calculating conditional expectations. You should take careful note of these, as we will be using them again and again.

What is the connection between the conditional expectation and the classical expectation? Our first rule clarifies this.

[8] For those who would like to know more, in the last section we give recommendations for further reading.

Rule 1 (Classical Expectation) *At time $t = 0$, the conditional expectation and the classical expectation coincide,*

$$\mathrm{E}_0[\widetilde{X}] = \mathrm{E}[\widetilde{X}] \ .$$

The rule shows that the conditional expectation represents a generalization of the classical expectation, incorporating additional information as well as future points in time beyond the present moment.

The second rule concerns the linearity, which is supposedly known to our readers for the classical expectation.

Rule 2 (Linearity) *For any real numbers a, b and any random variables $\widetilde{X}$ and $\widetilde{Y}$ the following equation applies*

$$\mathrm{E}_t[a\widetilde{X} + b\widetilde{Y}] = a\,\mathrm{E}_t[\widetilde{X}] + b\,\mathrm{E}_t[\widetilde{Y}] \ .$$

For technical reasons we need a further rule, which concerns real numbers. We know, that these quantities correspond to their expectations. That should now also be true when we are working with conditional expectations.

Rule 3 (Certain Quantity) *For the certain quantity 1 we have*

$$\mathrm{E}_t[1] = 1 \ .$$

An immediate conclusion from this rule affects all risk-free quantities. By linearity (Rule 2), the following is valid for real numbers X,

$$\mathrm{E}_t[X] = X\,\mathrm{E}_t[1] = X \ . \tag{1.5}$$

It should be kept in mind that the expectation E_t is not dealing with the information, which we will in fact have at this time. Much rather, it is dealing with the information that we are alleging at time t. The fourth rule makes use of our idea that as time advances, we get smarter.

Rule 4 (Iterated Expectation) *For $s \leq t$ it always applies*

$$\mathrm{E}_s \left[\mathrm{E}_t [\widetilde{X}] \right] = \mathrm{E}_s [\widetilde{X}] \ .$$

The rule underlines an important point of our methodology, although it is supposedly the hardest to understand. Nevertheless, there is a very plausible reason behind it. We repeatedly emphasized that we continually find ourselves in the present and are speaking only of our conceptions about the future. The actual development in contrast is not the subject of our observations. Rule 4 (Iterated Expectations) illustrates that.

Let us look at our knowledge at time s. If s comes before t, this knowledge comprises the knowledge that is already accessible to us today, but excludes beyond that the knowledge that we think we will have gained by time t. We should know more at time t than at time s: We are indeed operating under the idea, that as time goes on, we do not get more ignorant, but rather smarter. If our overall knowledge is meant to be consistent or rational, it would not make sense to have a completely different knowledge later on if it is built on our today's perceptions of the future.

If we wanted to verbally describe Rule 4 (Iterated Expectations), we would perhaps have to say the following: "If we today think about what we will know tomorrow about the day after tomorrow, we will only know what we today already believe to know tomorrow."

Rule 5 (Expectation of Realized Quantities) *If a quantity $\widetilde{X}$ is realized at time t, then for all other $\widetilde{Y}$*

$$\mathrm{E}_t [\widetilde{X}\widetilde{Y}] = \widetilde{X}\, \mathrm{E}_t [\widetilde{Y}] \ .$$

Rule 5 (Known Factor) fits our methodology of staying in the present while reasoning about the future. It illustrates the point-in-time principle once again: today, the quantity $\widetilde{X}$ is not (necessarily) a number but a random variable—at time 0 we can reason about $\widetilde{X}$, yet we do not know which value it will take at time t. At time t, by contrast, the realization of $\widetilde{X}$ is known given the information available then and $\widetilde{X}$ can be treated like a constant. Consequently, when we form the (conditional) expectation of the product $\widetilde{X}\widetilde{Y}$, we can take $\widetilde{X}$ out of the expectation—just as we do with ordinary numbers. In other words, Rule 5 (Known Factor) states that whenever a quantity is already known at time t, it can be treated as a certain number inside a conditional expectation taken at time t.

1.3.3 Examples and Problems

To make our discussion of conditional expectations and their properties more accessible, we will examine two illustrative cases. These are presented in the "Examples" sections of this book—one with a finite time horizon and the other with an infinite horizon. Both examples used throughout the book share an important feature: key characteristics, such as the stochastic structure of the unlevered cash flows, remain constant across sections, while only the financial configuration (insolvency risk, debt structure, etc.) varies. In addition, both cases are designed so that each node is followed by exactly two possible outcomes. We therefore refer to these examples as *binomial models*.

1.3.3.1 The Finite Case

We are dealing here with a company, which will bear three years of payments to its owners. The payments, however, cannot be predicted with certainty. Concentrate on Fig. 1.2.

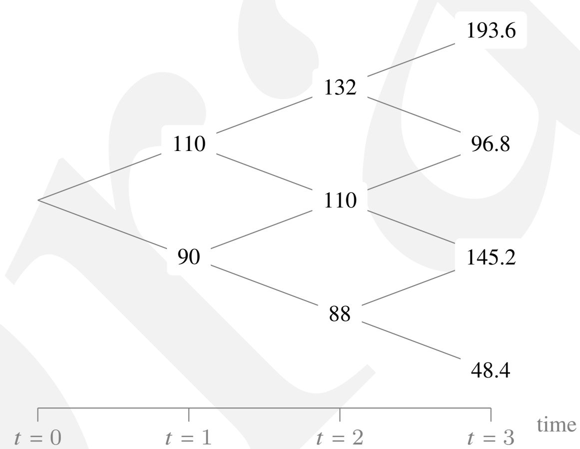

Fig. 1.2 Cash flows in the finite example.

In what follows, we must distinguish between two things. First, we refer to the movement that occurs immediately at a given node. Here, there are only two possibilities: "up" or "down" (binomial model). From now on we will be using the terms up and down not with respect to the actual cash flow movements. Rather, up means that the cash flow of the next period is stated at the top of the original value in our figure. Hence, up does not mean that the cash flow increases (as, for example, is clearly visible from $t = 2$ to $t = 3$ with the cash flow 110 going up to 96.8).

For any elementary movement up or down, we will use several terms, such as development, case, or movement. Second, we also consider the sequence obtained by combining these individual movements as they occur over time—for instance, a sequence such as up–down–up. We also use a shortened notation in which we keep only the initial letters, e.g., udu. Such a sequence will always be referred to as a "state" and denoted by ω; sometimes we also use the term "node" for convenience.

Although our language is occasionally somewhat informal, the distinction between states and the variables realized in them is important to keep in mind. The cash flow of 110 at $t = 2$ can be reached in two distinct ways: ud or du. Mathematically, these represent two different states, even though the resulting cash flow is identical. In this sense, 110 is not a state at $t = 2$; ud and du are.

We will later also consider cases in which one examines what follows a given state. Take, for example, $\omega = ud$. At the next point in time, the movements u and d are possible, and we will denote the resulting states by ωu and ωd.

In the following example, the probabilities of the states will become important. Again, one must be clear about what exactly is meant. For instance, $P(\omega u)$ denotes the probability that the path ω is realized first and is then followed by an up–movement.[9] Since the individual movements will be independent of one another and each occurs with probability 50%, this implies, for example, that for $\omega = u$,

$$P(uu) = 0.5 \times 0.5 = 0.25\ .$$

By contrast, $P(u|u)$ denotes something different: namely the conditional probability that, starting from the node $\omega = u$, an up–movement occurs. The symbol "|" indicates conditioning on the state, under which the probabilities of the upward movement, u, is evaluated. This value is, as mentioned already,

$$P(u|u) = 0.5\ . \tag{1.6}$$

1. In the first year ($t = 1$), two cases are conceivable, which we want to designate as up or down, as the case may be. If the development is up, then the payment amounts to 110, otherwise to 90. Both payments are equally possible.[10]
2. In the second year ($t = 2$), two additional possibilities emerge.
   - If we operate in $t = 1$ from up, the development can again proceed up or down. Should chance see to it that the sequence is up–up (abbreviated uu), then the cash flow in the second year amounts to 132. If the development goes in the opposite direction up–down (ud), then the owners only get a payment of 110.
   - But if the movement in $t = 1$ was down, then things could either turn up, or they could again be down. The development down–up (ud) leads at time $t = 2$ to payments in the amount of 110, in the case of down–down (dd), the owners only receive 88.

[9] In Sect. 1.2.3, we used the symbol $P_{t,t+1}$ to denote the price of a single future cash flow. This notation appears only there. Hence, following the common convention in the literature, we will from now on also use P to denote subjective probabilities. Unlike prices in Sect. 1.2.3, probabilities do not carry a double index $t, t+1$; at most, they carry a single time index indicating the date at which the probability is assessed.

[10] In case the actual development turns out to be neither 90 nor 110 our model was wrongly specified. Any decision based on this wrong model will be misleading. See our remark on the point-in-time principle in Sect. 1.2.4.

The three payment amounts can also be described differently: Chance twice ensures an unforeseeable development.

- If it goes up twice, then the cash flow amounts to 132. The probability for this comes to $0.5^2 = 25\%$.
- If it, in contrast, only goes up once, then the owners get a payment in the amount of 110. Since there are two ways of ending up with this payment, their probability comes to $2 \times 0.5 \times (1 - 0.5) = 50\%$.
- If, finally, it never goes up, then there is a cash flow of only 88, and the probability for that is $(1 - 0.5)^2 = 25\%$.

3. In the last year ($t = 3$) the following cases are possible, which can be stated as follows:

- The development proceeds up three times. That leads to a cash flow of 193.6, and namely with a probability of $0.5^3 = 12.5\%$.
- It only goes up twice.[11] There are three possible ways for moving to the payment of 96.8, which is why the possibility for this is $3 \times 0.5^2 \times (1 - 0.5)^1 = 37.5\%$.
- The development is now only up once. There are also three ways of reaching this amount, which is why its probability is $3 \times 0.5^1 \times (1 - 0.5)^2 = 37.5\%$, and namely with a payment in the amount of 145.2.
- Finally, it can also always go down. That leads to a cash flow of 48.4, and namely with a probability of $(1 - 0.5)^3 = 12.5\%$.

We will now determine the conditional expectations for every period and in so doing check our rules. As an example let us concentrate on the third-year cash flow and the conceptions, that we will probably have of it at time $t = 1$. It is thus dealing with the expression

$$\mathrm{E}_1[\widetilde{CF}_3] \ .$$

At time $t = 1$, two possibilities can have entered in. Let us at first look at the case that we had an up development. From here, three developments are conceivable, namely

- twice up: cash flow 193.6 with probability 25%,
- once up: cash flow 96.8 with probability 50%,
- never up: cash flow 145.2 with probability 25%.

The conditional expectation for the up condition at time $t = 1$ is then

$$0.25 \times 193.6 + 0.5 \times 96.8 + 0.25 \times 145.2 = 133.1 \ .$$

Now, we still have the case to look at that at time $t = 1$ we experienced a down development. Three developments are again conceivable from here that by an analogous procedure lead to

$$0.25 \times 96.8 + 0.5 \times 145.2 + 0.25 \times 48.4 = 108.9 \ .$$

From that we end up with

[11] As mentioned already, we continue to use the term "up" even if the path is $udu = (110, 110, 96.8)$ and the cash flow in this case in particular goes down from $t = 2$ to $t = 3$.

$$\mathrm{E}_1[\widetilde{CF}_3] = \begin{cases} 133.1, & \text{if the development in } t = 1 \text{ is up,} \\ 108.9, & \text{if the development in } t = 1 \text{ is down.} \end{cases}$$

At this point we discover that $\mathrm{E}_1[\widetilde{CF}_3]$ is a random variable, since at time $t = 0$ it still cannot be known which of the two conditions will manifest at time $t = 1$.

Let us now direct our attention to the unconditional expectation of the random variables $\widetilde{CF}_3$. We can get it directly from

$$0.125 \times 193.6 + 0.375 \times 96.8 + 0.375 \times 145.2 + 0.125 \times 48.4 = 121$$

It is, moreover, possible to determine the expectation of the random variables $\mathrm{E}_1\left[\widetilde{CF}_3\right]$. In order to show that we come to the same result this way, we again have to use our rules. But, we can limit ourselves here to the iterated expectation (Rule 4, Iterated Expectations) as well as the classical expectation (Rule 1, Classical Expectation). The following must apply

$$\begin{aligned} \mathrm{E}\left[\mathrm{E}_1[\widetilde{CF}_3]\right] &= \mathrm{E}_0[\mathrm{E}_1[\widetilde{CF}_3]] && \text{Rule 1} \\ &= \mathrm{E}_0[\widetilde{CF}_3] && \text{Rule 4} \\ &= \mathrm{E}[\widetilde{CF}_3]\ . && \text{Rule 1} \end{aligned}$$

And the result is

$$0.5 \times 133.1 + 0.5 \times 108.9 = 121\ .$$

The attentive reader will notice that the conditional expectation of the cash flow $\widetilde{CF}_3$ and that of $\widetilde{CF}_1$ are closely related. In our example, $\mathrm{E}_1[\widetilde{CF}_3] = 1.1^2\,\widetilde{CF}_1$ since $1.1^2 \times 110 = 133.1$ and $1.1^2 \times 90 = 108.9$. This property does not only apply to $t = 2$ and $t = 3$, but also to $t = 1$. It is not chosen arbitrarily and will be analyzed more thoroughly in Sect. 2.1.2. With this, we conclude our examination of the rules using the payment example.

1.3.3.2 The Infinite Case

An essential difference from our example that we have been looking at until now is that the firm will now live on indefinitely. We want to suppose that the cash flows follow a binomial process according to Fig. 1.3. Seen from time t onward, the cash flows can move up through time $t + 1$ with either factor u or move down with factor d, in whereby we will speak of an upward movement in the first case and a downward movement in the second case.

For two consecutive times, the following is always valid

$$\widetilde{CF}_{t+1} = \begin{cases} u\,\widetilde{CF}_t & \text{if the development in } t = 1 \text{ is up,} \\ d\,\widetilde{CF}_t & \text{if the development in } t = 1 \text{ is down.} \end{cases}$$

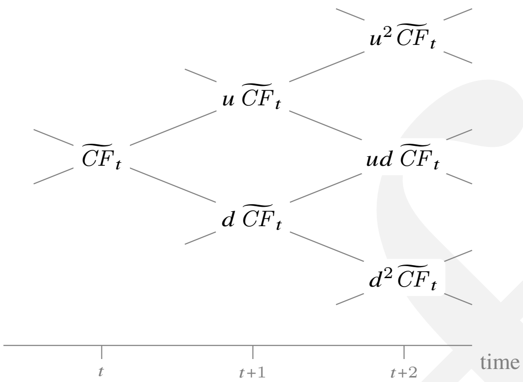

Fig. 1.3 Cash flows in the infinite example.

We will specify neither u nor d in more detail. We only suppose that they are not dependent upon time, positive and $u > d$.[12] As before, conditional probabilities at state ω and time $t+1$ with which the upward and downward movement occur will be denoted by $P_{t+1}(u|\omega)$ and $P_{t+1}(d|\omega)$. In the infinite example, these probabilities do not depend on time $t+1$ and node ω, hence, we omit these arguments in the notation from now on.

If under the given assumptions we determine the conditional expectation of a cash flow $\widetilde{CF}_{t+1}$ at time t, then the following applies

$$\begin{aligned} \mathrm{E}_t\left[\widetilde{CF}_{t+1}\right] &= P_{t+1}(u|\omega)\,u\,\widetilde{CF}_t + P_{t+1}(d|\omega)\,d\,\widetilde{CF}_t \\ &= \underbrace{\big(u\,P(u) + d\,P(d)\big)}_{=:1+g}\,\widetilde{CF}_t\ . \end{aligned}$$

The parameter g is thus not dependent on time. Out of this we can immediately derive the relation

$$\begin{aligned} \mathrm{E}_s\left[\widetilde{CF}_t\right] &= \mathrm{E}_s\left[\ldots\,\mathrm{E}_{t-1}\left[\widetilde{CF}_t\right]\,\ldots\right] \\ &= (1+g)\ldots(1+g)\,\widetilde{CF}_s \\ &= (1+g)^{t-s}\,\widetilde{CF}_s \end{aligned} \tag{1.7}$$

for $s \le t$ on the basis of Rule 4 (Iterated Expectations). We will always, for the subsequent examples, be working from the basis $CF_0 = \mathrm{E}[\widetilde{CF}_1] = 100$ and assume that all parameters have been so chosen that $g = 0$ is valid.

1.3.3.3 Problems

Problem 1.2 Look at the example in Fig. 1.4. Assume that the cash flows satisfy

[12] The factor u need not exceed one; it may be below one as long as $u > d$.

$$\widetilde{CF}_t = \mathrm{E}_t\left[\widetilde{CF}_{t+1}\right] .$$

(This particular property of cash flows will become important in the next chapter.) Assume that the up and down movements occur with the same probability. Fill in the gaps.

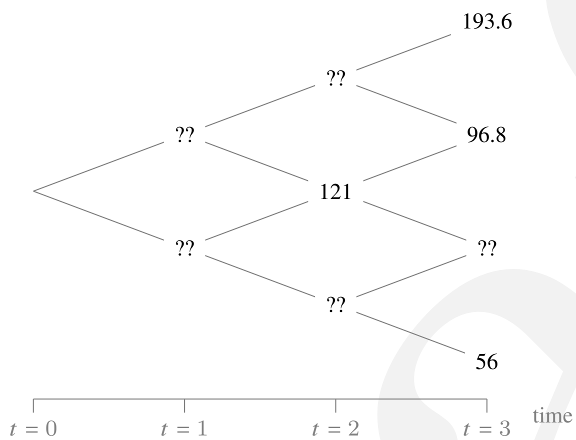

Fig. 1.4 Cash flows in Prob.1.2.

Problem 1.3 The following problem shows that expectations under different probabilities cannot be changed arbitrarily.

Go back to the solution of the last problem (see again Fig. 1.4). Let there be two different probabilities: P assigning the same probability to the up and down movements and Q that assigns 0.1 to the up and 0.9 to the down movement. Verify that

$$\mathrm{E}_0\left[\mathrm{E}_1^Q\left[\widetilde{CF}_2\right]\right] = \mathrm{E}_0^Q\left[\mathrm{E}_1\left[\widetilde{CF}_2\right]\right] .$$

This particular property of probabilities will be necessary when showing that cost of capital are also discount rates. But be careful if the binomial tree distinguishes between up–down and down–up as shown in Fig. 1.5. Verify that in this case the following terms are not equal,

$$\mathrm{E}_0\left[\mathrm{E}_1^Q\left[\widetilde{CF}_2\right]\right] \neq \mathrm{E}_0^Q\left[\mathrm{E}_1\left[\widetilde{CF}_2\right]\right] .$$

Problem 1.4 The following problem shows that for arbitrary but constant expectations of cash flows

$$\mathrm{E}\left[\widetilde{CF}_1\right] = \mathrm{E}\left[\widetilde{CF}_2\right] = \mathrm{E}\left[\widetilde{CF}_3\right] = CF_0$$

a binomial tree can be established where the conditional expectations of these cash flows satisfy an assumption that was mentioned in Prob. 1.2.

Consider a binomial tree where up and down movements occur with the same probability. Let two arbitrary numbers u, d be given such that (this will ensure $g = 0$)

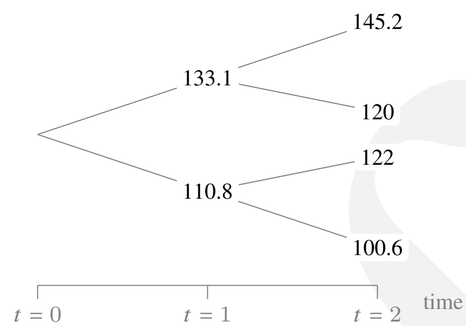

Fig. 1.5 Cash flows in Prob.1.3.

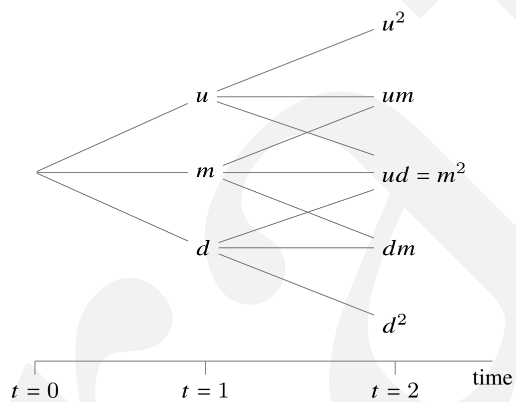

Fig. 1.6 Cash flows (trinomial tree) in Prob.1.5.

$$\frac{1}{2}u + \frac{1}{2}d = 1, \qquad u > d > 0.$$

Show that the cash flows following the binomial tree in Fig. 1.3 satisfy

$$\mathrm{E}_2\left[\widetilde{CF}_3\right] = \widetilde{CF}_2, \quad \mathrm{E}_1\left[\widetilde{CF}_2\right] = \widetilde{CF}_1, \quad \mathrm{E}_0\left[\widetilde{CF}_1\right] = CF_0 \ .$$

Problem 1.5 Consider a trinomial tree as in Fig. 1.6. Three movements are possible: "up", "middle" and "down" (we use u, m and d as in Fig. 1.6). They occur with the same probability and furthermore (so that the tree is recombining) ud yields the same as mm

$$u\,d = m^2 \ .$$

What further assumptions on u, m, d are necessary so that in this example

$$\mathrm{E}_1\left[\widetilde{CF}_2\right] = \widetilde{CF}_1$$

holds?

1.4 A First Glance at Business Values

In the following section, we introduce a generalized form of a business valuation concept that will be used throughout this book. The relationships between firm value and the cost of capital discussed here are to be understood in a general sense, as we wish and are still able to leave open whether we are analyzing levered or unlevered firms. For this reason, our exposition does not yet require a fully refined notation.

1.4.1 Valuation Concept

Value and Price Anyone addressing methodological questions of company valuation is well advised to carefully distinguish between the concepts of *value* and *price*. In financial theory, a clear separation of these two terms is often not made; instead, the term value is frequently used. We will also treat value and price as synonyms in this book. However, there are areas in economics where it is very important to distinguish these concepts precisely. Therefore, we will briefly explain what differentiates the two terms from each other.

Prices refer to historical and current transaction amounts; such prices can be observed easily and do not require additional assumptions. On the other hand, we speak of values when referring to future transactions. Some points need to be addressed here:

– Unlike prices, we cannot know what we will observe in the future when it comes to values. We have already pointed this out: values are uncertain. Here, we aim to make reasonable assumptions that allow us to analyze this uncertainty.
– It is immediately clear that a calculation or a theory is necessary to explain how an analyst should arrive at values.[13] No theory, no values. In this book, for example, we introduce the theory of discounted cash flow. And every theory is based on assumptions; this is also not different for a theory of business valuation. Anyone wishing to apply the theory must therefore examine whether the assumptions are plausible, logically consistent, and whether empirical evidence supports its conclusions.
– Finally, it should be noted that values, in contrast to prices, will be subjective. When different people determine prices, they must inevitably arrive at the same result. This is not the case for values. Assumptions are always subjective: for example, the number of states, the assumed cash flows, or the probabilities of occurrence are subject to personal judgment. This is also true in the theory of discounted cash flow. Even when using the DCF calculation, one cannot assume that two individuals will arrive at exactly the same valuation result.

In what follows, we will (as noted above) use the same symbol, V_t, to denote both prices (when $t = 0$ refers to the present) and values (when $t > 0$ refers to the future).

[13] That at this point (for example in the Capital Asset Pricing Model) people often speak of *price theory* rather than *value theory* does not necessarily facilitate understanding. You do not need a theory when dealing with past prices.

No Arbitrage and Equilibrium Trade has existed since time immemorial, and even the earliest written records discovered by humankind concern the prices of goods. Economic theory aims to explain why prices are what we observe. It is fair to say that today the school of neoclassicism has become dominant. For example, while Marxism attributes prices to the amount of labor contained in goods, neoclassicism determines prices solely by the interplay of supply and demand.[14]

Anyone who wants to know exactly how the price of a particular good or service is determined will find two related concepts in neoclassicism: the idea of equilibrium and the no-arbitrage principle. We will briefly discuss the interaction of these two approaches.

First of all, it can be stated that in any equilibrium, arbitrage opportunities must be absent. On the other hand, the absence of arbitrage does not necessarily mean that we have already found an equilibrium. In this sense, the no-arbitrage condition is the logically weaker concept; it requires fewer restrictive assumptions. In earlier versions of our book, we were tempted to give preference to models that rely solely on no-arbitrage conditions because their assumptions are harder to criticize. We no longer regard this perspective as valid, as it is didactically misleading and fails to recognize the importance of the equilibrium concept. This is because equilibrium and no-arbitrage refer to two different things that are nevertheless related:

– If one wants to explain how prices can be derived from so-called fundamental data (elements that themselves are not prices), one must use an equilibrium theory. Equilibrium relates a price to supply and demand and, in specific models, even allows for calculating the exact numerical value of a price.
  It must be understood that this theory does not attempt to describe how a price "comes into being" in a market or how market participants agree on this price. In physics, it is also not explained how gravity causes Earth to attract a stone; rather, gravity is assumed and described in terms of its strength—then one can determine from this how fast the stone moves toward the ground. The question "what makes gravity act on a stone" makes no sense in physics.
  Similarly, equilibrium theory argues that instead of thinking about how prices form in equilibrium, it assumes that supply and demand are in balance, and that observed prices emerge from this togetherness. Therefore, equilibrium and prices cannot be separated conceptually.
– The arbitrage theory follows a different approach. It does not aim to explain prices based on fundamental data but rather asks whether knowledge of given prices allows conclusions about what *other* prices should be—that is, whether particular prices for some objects imply particular prices for others. Here, it becomes clear that prices are not explained but rather derived from other prices. One could say that arbitrage theory builds on equilibrium theory, even though it has its own distinctive features.

Our considerations in this book are characterized by focusing exclusively on arbitrage theory. Our main focus will always be on deriving prices for other securities based on known prices—this characterizes the theory of stochastic discounted cash flow.

[14] Of course, this presupposes that one can describe what is meant by supply and demand. For the latter, utility functions come into play.

If we assume that the capital market does not allow for a free lunch we are quite sure that this assumption is indisputable. One is usually convinced that arbitrage opportunities can hold only briefly and will be exploited sooner or later so that they disappear. For us, this assumption is straightforward.

Assumption 1.1 (No free lunch) *The capital market is free of arbitrage.*

This assumption is typically specified in finance with the help of an extravagant formalism. We would like to do away with that here, since the effort far outweighs the usefulness and the following does not require such details. We limit ourselves to illustrating the assumption. Arbitrage free means, in loose terms, that no market participant is in the position to make earnings from nothing. Anyone, who has some cash inflows will put up with cash outflows. We will go into the conclusions of this assumption in somewhat more detail.

The central building block of our theory of business valuation is the so-called *Fundamental Theorem of Asset Pricing*. It can be derived from the concept of no arbitrage. We will not go through the trouble of proving the theorem. But to at least make it plausible, we introduce alternative valuation concepts. While doing so, we limit ourselves for simplicity's sake to the one-period case.

The risk-free interest rate in this book does not carry a time index t. By doing so, we implicitly assume that risk-free rates are constant over time. This restriction is not essential and can easily be relaxed: if the interest rate varies over time, it simply takes on a time index, and the propositions of our theorems remain valid. We adopt this convention solely for the sake of clarity.

Valuation (in the One-Period Case) under Certainty Imagine a capital market in which only claims for deterministic payments are traded that are due in $t = 1$. You can think of a firm, which pays dividends at time $t = 1$ in the amount of CF_1. It is obvious how these securities are to be valued when the capital market is arbitrage free. The deterministic cash flows are discounted at the risk-free interest rate. If we denote the interest rate with r_f, then equation

$$V_0 = \frac{CF_1}{1 + r_f} \tag{1.8}$$

must be valid. Otherwise, one could end up with a strategy that practically comes down to operating a private money pump. And that would be at odds with the no-free-lunch assumption.

If, for instance, the left hand side of the equation were smaller than the right hand side, an investor could take out a loan today with the interest rate r_f and acquire the firm for the price V_0. The sum of all payments at time $t = 0$ would amount to zero. At time $t = 1$, she would retain the cash flows and pay back the loan. And the now remaining account balance would presumably be positive,

$$CF_1 - \left(1 + r_f\right) V_0 > 0 \ .$$

That is an arbitrage opportunity, and it is exactly such results that we would like to exclude with the Assump. 1.1.[15]

Valuation (in the One-Period Case) under Uncertainty Future payments are uncertain in the world we consider. We must therefore ask in what way the relationship described in the Equation (1.8) can be generalized. For simplicity, we assume that at $t = 1$ the cash flow can take two possible values, which we denote by up and down, corresponding to payments of 110 and 90, respectively. There are three different ways, the last of which deserves our particular attention, to incorporate this risk into valuation.

Certainty Equivalent If the cash flows payable at time $t = 1$ are stochastic, one may ask which risk-free payment at time $t = 1$ the investor finds just as attractive as the risky cash flows. Such a payment is called a certainty equivalent. The certainty equivalent will be discounted at the risk-free rate, instead of the expected payments at time $t = 1$. Assume further, that the two relevant states of nature are equally probable and the risk-free interest rate is $r_f = 5\%$. The certainty equivalent itself must be determined with the help of a utility function. According to our knowledge, this formulation is rather seldom incurred in practice.

To illustrate the procedure through an example, let us again look at our firm. We know

$$\widetilde{CF}_1 = \begin{cases} 110 & \text{if the development in } t = 1 \text{ is up,} \\ 90 & \text{if the development in } t = 1 \text{ is down.} \end{cases}$$

The certainty equivalent is now that payment S_1 that is just as attractive as the lottery $\widetilde{CF}_1$, in which one gets 110 or 90 respectively with the same probability. If the investor has an expected utility representation $u(\cdot)$, that corresponds to equation

$$u(S_1) = \mathrm{E}\left[u\left(\widetilde{CF}_1\right)\right] \ ,$$

from which, according to the application of the reverse function, we get

$$S_1 = u^{-1}\left(\mathrm{E}\left[u\left(\widetilde{CF}_1\right)\right]\right) .$$

If, for example, $u(x) = \sqrt{x}$ is the utility function, then the certainty equivalent in our example amounts to

$$S_1 = \left(0.5 \times \sqrt{110} + 0.5 \times \sqrt{90}\right)^2 \approx 99.75 \ .$$

From that we finally get,

$$V_0 = \frac{S_1}{1 + r_f} \approx \frac{99.75}{1 + 0.05} = 95.00 \ .$$

[15] We could make a completely analogous argumentation, if the left hand side of Eq. (1.8) were larger than the right hand side.

Cost of Capital There is the much more widely spread technique in the valuation practice of raising the risk-free interest rate with a risk premium. The sum of both quantities is the cost of capital. In Eq. (1.8) we then adjust the denominator and not the numerator.

If the risk premium is named z, then the value of the firm in our example results from

$$V_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{1 + r_f + z} ,$$

and one would come to the same result using the relevant numbers here with a risk premium of $z \approx 0.264\%$ as with the certainty equivalent method, since

$$V_t \approx \frac{0.5 \times 110 + 0.5 \times 90}{1 + 0.05 + 0.00264} = \frac{100}{1.05264} = 95.00 .$$

Risk-Neutral Probabilities In this third formulation, again the risk-free interest rate r_f is used as the discount rate. In the numerator we find an expectation of the risky cash flow. But the expectation is not calculated with the actual estimated subjective probabilities. Instead of that, risk-adjusted probabilities are used, which are also called risk-neutral.

In our example the subjective probabilities for the up, or, as the case may be, down development come to 50% in each case. It is plain to see that one can express risk aversion with measuring up with a lesser weight and down with a heavier weight.

If the expectation calculated on the basis of risk-neutral probabilities is denoted by $\mathrm{E}^Q[\cdot]$, then the valuation formula in our example runs

$$V_0 = \frac{\mathrm{E}^Q\left[\widetilde{CF}_1\right]}{1 + r_f} .$$

It can be shown that with the relevant numbers here a risk-neutral probability of 48.75% for up leads to the same result as both concepts already described, since

$$V_0 \approx \frac{0.4875 \times 110 + 0.5125 \times 90}{1 + 0.05} = \frac{99.75}{1.05} = 95.00 .$$

Several different terms have been used in the literature for this third way. Equivalent martingale measure is also often spoken of, instead of risk-neutral probability.

We now turn to the second approach in more detail and will then be dealing with the last approach.

1.4.2 Cost of Capital as Conditional Expected Returns

Discount Rates and Expected Returns To obtain a preliminary notion of the cost of capital under uncertainty, let us take a look at the textbooks. It is often said that cost of capital is an expected return. For example, Copeland et al. (2005, p. 557) use the expression "rate of return" instead of "cost of capital," in order to present the valuation concept for a firm. Brealey et al. (2025, p. 9-2) write explicitly, that the "assets cost of capital . . . is the return that investors require for holding the firm's assets and bearing the risk associated with doing so." de Matos (2001, p. 42f.) also explains that the cost of capital involves expected returns.

However, we also then find the suggestion within the literature to bring the notion of the cost of capital to life somewhat differently. This concerns the idea that the cost of capital should be suitable as discount rate for future cash flows. Brealey et al. (2025, p. 9-9), for instance, speak of the cost of capital as those figures with which cash flows are to be discounted. There is a related remark in Miles and Ezzell (1980, p. 722), for example, where it is said that "at any time k, ρ is the appropriate rate for discounting the time i expected unlevered cash flow in period j where ρ is referred to as the unlevered cost of capital."

It is not recommended to simply equate expected returns and discount rates. This warning does not easily stand to reason with the critical reader. This will, however, change from now on.

Conditional Expected Returns How can our definition of the cost of capital be generalized, if the future is uncertain? We shift to time $t > 0$. The analyst will have gathered some knowledge about the returns at this point in time. In addition, she sets the return flow of payments one period later in proportion to the capital employed. When the analyst does this based on the information available at time t, the capital employed $\widetilde{V}_t$ is known with certainty to her. The analyst can now treat the invested capital like a real number, granted that she always assumes the information at time t. The analyst considers the conditional expectation of the repayment in $t+1$ against the value of the capital $\widetilde{V}_t$. Through this understanding of the term cost of capital, a derivation of the valuation equation analogous to (1.1) will be possible. Hence, we regard the following definition of the cost of capital as appropriate.

Definition 1.1 (The firm's cost of capital) *The cost of capital $\widetilde{k}_t$ of a firm is the conditional expected return*

$$\widetilde{k}_t := \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{\widetilde{V}_t} - 1 \,.$$

Before we move on, we must, however, take note that our cost of capital definition has a possible disadvantage. Look at numerators and denominators separately. In the numerator of Def. 1.1, the expectations of payments at time $t+1$ stand under the condition that the analyst possesses the information of time t. One cannot simply take these expectations as given. It is much more likely that these conditional expectations at time t themselves are random variables. Dividing these random variables by $\widetilde{V}_t$ yields another random variable, regardless of whether $\widetilde{V}_t$ is deterministic or stochastic. But that means that the cost of capital, as it is so defined, constitutes a random variable. Future expected returns on both equity and debt thus become random variables—something that will come as no surprise to practitioners. Unfortunately for the theorist, one cannot discount today using quantities whose realization is not yet known at time $t = 0$. So, we have a definition of the cost of capital that is not suitable for our intended purpose here!

Deterministic Cost of Capital We will escape this dilemma only by making a heroic assumption. Specifically, we assume that the cost of capital, as defined in Definition 1.1, is deterministic. In other words, we simply take the cost of capital to be known with certainty and will later show that it indeed serves as an appropriate discount rate. In other words, those who view the cost of capital as a conditional expected return and assume it to be deterministic may also use it as a discount rate. [16]

Our critical colleagues may want to object at this point that the knowledge of future anticipated returns in the valuation of firms depicts all too heroic of an assumption; one which cannot actually be met in real life. To that, we can only answer that it does indeed involve a very large assumption. Only, without this assumption, no one, not even our critics, can prove a valuation equation analogous to Eq. (1.1). Thus seen, although the assumption is admittedly heroic, it is also indispensable for a theory of business valuation. Those who principally reject it, must also renounce the determination of firms' market values through the use of a DCF approach. Regrettably, we have no other choice in the matter.

The attentive reader will still recall the objection mentioned in the previous section. We had noticed that "time will tell." If we already assume at time $t = 0$ that the cost of capital is known with certainty, we are implicitly assuming that a vast amount of information has already been acquired. Everything that can be ascertained about cost of capital is already known today. At this point we repeat the statement that there is no business to be done without the assumption of deterministic cost of capital.

Attempts at Other Definitions? We would like to emphasize, as clearly as possible, that our definition of the cost of capital cannot be simplified any further. At first glance, there seems to be nothing wrong with adopting Eq. (1.3) and interpreting the introduction of uncertainty through an (unconditional) expectation. The expression

$$k_t \stackrel{?}{=} \mathrm{E}\left[\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t} - 1\right]$$

would then describe a cost of capital.

As an academic, you have the freedom to choose your terms as you like. Definitions of cost of capital can be neither true, nor false. They are at best suitable, or unsuitable. And the preceding definition is by all means unsuitable. To allow for an equation of the form

[16] We are not the first to point this out: "As in the single cash flow example of preceding sections, a capital market where prices each period are set according to the Sharpe-Lintner-Black model in general requires that there are non-stochastic discount rates. . . " Fama (1977, p. 18).

$$V_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_2\right]}{(1+k)^2} + \dots$$

to be obtained it is necessary to mathematically "detach" $\widetilde{V}_t$ from the expectation above. But this is impossible for a random variable due to Jensen's inequality.[17]

Even a more refined approach, such as in equation

$$k_t \stackrel{?}{=} \frac{\mathrm{E}\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{\mathrm{E}\left[\widetilde{V}_t\right]} - 1$$

does not lead to the kind of results with which we can be satisfied. True, if k_t is constant through time this is easily converted to the following important equation

$$V_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_2\right]}{(1+k)^2} + \dots .$$

But we had more in mind for our definition of cost of capital. We were not just thinking about a computational rule, which allows for the current business value to be determined. More importantly, it should be possible to determine future business values. While calculation of V_0 can now be managed with this definition of cost of capital, we cannot, however, get an equation in the form of

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}\right]}{1+k} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1+k)^2} + \dots$$

from the definition, since the cost of capital always depicts unconditional expectations. This definition of cost of capital also turns out to be unsuitable for our purposes. Thus, we have no choice but to assume the following.

Assumption 1.2 (Deterministic cost of capital) *The cost of capital k_t of a firm are non-random (deterministic).*

Let us summarize. With a DCF approach, cost of capital is, sensibly enough, taken as conditional expected returns. This idea will be the red thread throughout our presentation. As already stated, we will still assume in the following that the cost of capital does not involve random, but rather deterministic quantities. We cannot get any valuation

[17] For a random variable $\widetilde{V}_t$ that is not deterministic

$$\frac{1}{\mathrm{E}\left[\widetilde{V}_t\right]} \neq \mathrm{E}\left[\frac{1}{\widetilde{V}_t}\right]$$

holds due to Jensen's inequality. A similar result will apply to our attempted definition of cost of capital.

equation without this assumption.[18] On the basis of this assumption, we shall now derive valuation equations. In line with Milton Friedman's understanding of science we will then examine whether our findings obtained withstand empirical examination. But first, let us establish some results.

1.4.3 A First Valuation Equation

In order to derive the valuation equation, we will use our rules for conditional expectations for the first time. Although this calculation is rather simple, we want to deal with it in some detail. Our intention in doing so is to get our readers more comfortable with the formal usage of the rules in Sect. 1.3.2.

Theorem 1.1 (Market value of the firm) *When the cost of capital k_t is deterministic, then the firm's value at time t amounts to*

$$\widetilde{V}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s\right]}{(1+k_t)\dots(1+k_{s-1})} .$$

The reader used to the product symbol Π might prefer the expression

$$\widetilde{V}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s\right]}{\prod_{v=t}^{s-1}(1+k_v)}$$

as more compact.

To prove this statement, let us reformulate Def. 1.1 of the cost of capital to

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1+k_t} .$$

The above equation is recursive, as $\widetilde{V}_t$ is a function of $\widetilde{V}_{t+1}$. If we use the appropriate relation between $\widetilde{V}_{t+1}$ and $\widetilde{V}_{t+2}$, the result is

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1} + \frac{\mathrm{E}_{t+1}\left[\widetilde{CF}_{t+2}+\widetilde{V}_{t+2}\right]}{1+k_{t+1}}\right]}{1+k_t} .$$

[18] This is accurate as far as it goes, but see Sect. 2.1.4 for further discussion on this topic.

Because the cost of capital is deterministic and the conditional expectation is linear, we can reformulate this expression with the help of Rule 2 (Linearity) to

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}\right]}{1+k_t} + \frac{\mathrm{E}_t\left[\mathrm{E}_{t+1}\left[\widetilde{CF}_{t+2}\right]\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}_t\left[\mathrm{E}_{t+1}\left[\widetilde{V}_{t+2}\right]\right]}{(1+k_t)(1+k_{t+1})} \ .$$

Rule 4 (Iterated Expectation) allows us to notate the expectation in a more simple form,

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}\right]}{1+k_t} + \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}_t\left[\widetilde{V}_{t+2}\right]}{(1+k_t)(1+k_{t+1})} \ .$$

If we continue this procedure through time T, we get

$$\widetilde{V}_t = \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}\right]}{1+k_t} + \ldots + \frac{\mathrm{E}_t\left[\widetilde{CF}_T\right]}{(1+k_t)\ldots(1+k_{T-1})} + \frac{\mathrm{E}_t\left[\widetilde{V}_T\right]}{(1+k_t)\ldots(1+k_{T-1})} \ .$$

If T denotes a (finite) terminal horizon, then setting $V_T = 0$ (a condition we shall refer to as transversality) eliminates the last term; a more detailed discussion of this assumption and the complications it entails for infinite horizons ($T \longrightarrow \infty$) will follow in Sect. 1.4.5. This gives us

$$\widetilde{V}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s\right]}{(1+k_t)\ldots(1+k_{s-1})} \ . \qquad \square$$

Under the prerequisite that we evaluate the firm at time $t = 0$, it follows from Thm. 1.1 together with Rule 1 (Classical Expectation)

$$V_0 = \sum_{s=1}^{T} \frac{\mathrm{E}\left[\widetilde{CF}_s\right]}{(1+k_0)\ldots(1+k_{s-1})}$$

and finally in the particular case of time-invariant cost of capital

$$V_0 = \sum_{t=1}^{T} \frac{\mathrm{E}\left[\widetilde{CF}_t\right]}{(1+k)^t} \ .$$

1.4.4 Fundamental Theorem of Asset Pricing

The idea of an equivalent martingale measure has to do with offering the analyst other probabilities for the states of nature and thus making her world risk-neutral. That this method is always successful, that there always exist probabilities in which the valuation can be made risk-neutral, is the content of the Fundamental Theorem of Asset Pricing.

The Fundamental Theorem maintains that the investor's subjective probability can be replaced by another probability measure Q. We denote the new expectations with E^Q. The following statement is valid for them.

Theorem 1.2 (Fundamental Theorem of Asset Pricing) *If the capital market is free of arbitrage, then conditional probabilities Q can be chosen to the extent that the following equation is valid,*

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + r_f} . \tag{1.9}$$

The Fundamental Theorem is valid for all conceivable financial claims, for equities as well as debts, for levered as well as unlevered firms.

The real challenge in Thm. 1.2 does not lie in showing that it is possible to select numbers $Q(\cdot)$ which, when inserted into an equation such as (1.9), match the prices observed in the market. Mathematically, this is merely a matter of rearranging the equation and solving for the corresponding values of $Q(\cdot)$; since the equation is linear, a solution in the real numbers will always exist. What is by no means trivial, however, is the assertion that the resulting values actually lie between 0 and 1. Demonstrating this is the true difficulty of the proof.

In Thm. 1.1, we were able to gain a preliminary equation for the valuation of firms out of the definition of the cost of capital. In a completely analogous way, we could show that with the help of risk-neutral probabilities a valuation equation can also be proven. It runs

$$\widetilde{V}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s\right]}{\left(1 + r_f\right)^{s-t}} , \tag{1.10}$$

and does not need a new proof.

If we compare the Fundamental Theorem 1.2 with Eq. (1.8), we then determine similarities and differences: here as there, we are discounting with the risk-free interest rate. Instead of deterministic payments, we now deal with their conditional expectations. The risk of future payments comes into the equation in so far as we are not working with the subjective probabilities for future states of nature, but rather instead with the so called risk-neutral probabilities. It deals with a third way, other than certainty equivalents and cost of capital, of incorporating risk into the valuation equation. Again and again throughout this book we will see that the Fundamental Theorem can be put to practical use.

Risk Neutrality Before we turn to the practical application of the concept, we want to answer the question as to why the probabilities used here are termed as "risk-neutral" whatsoever. We will make use here of our rules for conditional expectations.

For this purpose, we reformulate the Fundamental Theorem. It goes as

$$1 + r_f = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right]}{\widetilde{V}_t} \ .$$

We know according to Rule 5 (Known Factor), that the value of the firm at time t is already known and therefore may also be included within the conditional expectation, without changing the result,

$$1 + r_f = \mathrm{E}_t^Q\left[\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t}\right] \ .$$

The one on the left hand side is deterministic. By Rule 3 (Certainty), we can replace it with its expectation,

$$r_f = \mathrm{E}_t^Q\left[\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t}\right] - \mathrm{E}_t^Q\ [1] \ .$$

Lastly, we use Rule 2 (Linearity) and get

$$r_f = \mathrm{E}_t^Q\left[\frac{\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t} - 1\right] \ .$$

This presentation of the Fundamental Theorem allows for a clear economic interpretation. We look to the returns as an argument for the conditional expectation. An investor attains these if she acquires a security at time t, receives the cash flows one period later and immediately afterwards again sells the security. The statement of the last equation then reads that the conditional expected return of this strategy is always the risk-free rate, if the risk-neutral probabilities are fallen back upon instead of the subjective probabilities. The designation “risk-neutral probability” for Q relates back to this characteristic.

Uniqueness of *Q* The question arises whether the mentioned probability Q is unique and what consequences can be drawn if several risk-neutral probabilities exist. We do not want to go into detail here, but the following can be said. For the existence of the risk-neutral probability measure it is completely sufficient that the market is free of arbitrage. The assertion that any claim, so complicated as it may be, is also traded (in this case the market is called complete), is not necessary to prove the Fundamental Theorem of Asset Pricing. Admittedly, in this case of an incomplete market one can show that Q need not to be unique. But this is not a problem at least for valuation, since every “possible” probability will lead to one and the same price of the firm.

It is, however, a problem if the firm to be valued is not or not yet traded on the market. In this case the Fundamental Theorem may not yield one particular value but instead a range of possible prices of the firm. Therefore, a second assumption that is also necessary for DCF valuation is the following.

Assumption 1.3 (Spanning) *The cash flows of an asset to be valued can be perfectly duplicated at the capital market.*

Let us discuss this assumption. Again we suppress the formalism required to formulate it in a mathematically precise manner. The assumption requires that the possible cash flow of the company, complicated as it may be, can also be achieved by holding a (possibly involved) portfolio of stocks, bonds, derivatives or any other assets from the capital market. Instead buying the company, the investor could turn to the capital market and would receive not only a similar, but the same distribution of cash flows.

It might be questionable, why the investor at all would now buy the company if she can invest in a portfolio of equities and bonds. But we do not care for determining an investor's optimal portfolio (this would require an examination of her utility function). Instead we are decided to value a company by trying to replicate the company's cash flows by a portfolio of traded assets. We stress explicitly here that in this book valuation is a comparison of a company with the capital market. Spanning is necessary to perform this comparison.

At the beginning of Sect. 1.4.1, we pointed out that equilibrium and arbitrage are two neoclassical concepts for determining prices (values). While equilibrium relies on fundamental data, arbitrage theory justifies the price of an object by reference to the prices of other objects. This way of thinking becomes especially clear in Assump. 1.3. Put differently, no other assumption illustrates as clearly that DCF methods rest on an arbitrage argument.

Risk-Neutral Probability and Pricing Kernel In modern finance theory, it has become standard practice to work not with risk-neutral probabilities, but with the so-called pricing kernel or stochastic discount factor m_{t+1}. Equation (1.9) then takes the form

$$\widetilde{V}_t = \mathrm{E}_t\left[m_{t+1}\left(\widetilde{CF}_{t+1} + \widetilde{V}_{t+1}\right)\right] \ . \tag{1.11}$$

From an economic perspective, this formulation does not differ from the familiar representation based on risk-neutral probabilities.[19] Why, then, do we not employ the tool that has long been the standard in the theoretical literature?

We have three reasons for this choice:

1. We believe that risk-neutral probabilities lend themselves more readily to intuitive explanation. Admittedly, this is largely a matter of taste and may not persuade every reader.
2. In corporate valuation, the emphasis is typically on practical applications. Neither Q nor m_t are directly observable quantities, and therefore many important (and in some cases celebrated) theorems make no explicit use of them. The risk-neutral probability measure exists, but it remains primarily a theoretical construct designed to facilitate

[19] Mathematically, the two formulations are equivalent by virtue of the Radon–Nikodym theorem.

proofs. Our main concern, therefore, is didactic clarity: through which approach can these proofs be presented most transparently? We have found the path via Q to be the more accessible one.

3. Moreover, in the empirical literature, the pricing kernel is typically introduced as an object to be inferred, with the aim of obtaining quantitative estimates for a broad set of assets rather than valuations for a specific company. In contrast, the applied literature on corporate valuation rarely engages in such estimation exercises. Its aim is usually to value a specific firm rather than to propose a general procedure for valuing all stocks in the market.

1.4.5 Transversality and Infinite Lifespan

We are working under the assumption in any case that the value of the business closes in on zero, as the firm's end approaches. In the case of a firm with a finite life-span, that is a very obvious, even trivial, statement. Beyond time T, the cash flows do not flow any more, which is why the value of the firm must disappear. But we also need an analogous characteristic when dealing with a firm that has an infinite life span. It must be kept in mind that dealing with infinity is anything but straightforward; once the time horizon is extended without bound, what we commonly regard as common sense can be stretched to its limits. In the formally oriented literature, the so called "transversality" is spoken of when the discounted business value is heading towards zero with an infinite time-horizon.[20]

Transversality is not only a technical condition but can be interpreted in an economically meaningful way. Let us assume for a moment that transversality would not be satisfied. Then relying only on the Fundamental Theorem (1.9) will not yield a *unique* value $\widetilde{V}_t$ of the company.[21] Instead, there exists an infinite number of variables $\widetilde{V}_t$ that will satisfy the Fundamental Theorem and could be acknowledged as "the" value of the company. There is not one but an infinite number of such "values." Valuing a company using only the Fundamental Theorem is doomed to fail. Transversality is in this sense indispensable.

If transversality is to ensure that residual firm value disappears as the horizon tends to infinity, then the question arises of how this requirement should be expressed. It is by no means clear how to formulate the condition in a precise and satisfactory manner. In the Wiley edition of this book we indeed used the specification

$$\lim_{T\to\infty} \frac{\mathrm{E}\left[\widetilde{V}_T\right]}{(1+k)^T} = 0\ .$$

[20] The term transversality originates from differential topology, where it describes conditions ensuring that objects intersect in a "well-behaved" manner. In economics and finance, the notion was adopted to denote a boundary condition that rules out explosive or "bubble" solutions.

[21] For details see Kruschwitz and Löffler (2013, Theorem 2); an illustrative example is given in Prob.1.10.

Meanwhile, it has become clear that this approach leads nowhere. Moreover, a mathematical difficulty arises: the formulation above involves a limit. In the case of numbers such a limit is well defined, but in our setting the requirement stated above lacks a clear meaning. Because our model involves future firm values, we have to confront limits of random variables; yet this only deepens the difficulty, since it is far from clear what exactly a "limit" of random variables should mean.

Neither common sense nor intuition is a reliable guide—especially when conditional expectations of random variables are involved, which may themselves be random variables. Because random variables are functions, one must recognize that there is no single, universally valid definition of a limit for functions. Instead, there are several concepts at hand.[22] One can show that the so-called *almost-everywhere convergence* is appropriate.[23] On this basis we say that a sequence of random variables $\widetilde{X}_t$ converges to a random variable $\widetilde{X}$ if the following is true: For every state ω we ask whether

$$\lim_{T\to\infty} \widetilde{X}_t(\omega) = \widetilde{X}(\omega)$$

holds. By fixing the state ω we are no longer dealing with sequences of random variables, but instead with sequences of numbers. Based on this we can be certain whether a limit exists and equals a number $\widetilde{X}(\omega)$. However, we do not know how large the set of all such states ω is. Therefore we say that $\widetilde{X}_t$ *converges almost everywhere* to $\widetilde{X}$ if the set of such states has probability one. In plain terms: "The sequence $\widetilde{X}_t(\omega)$ converges to $\widetilde{X}(\omega)$ in almost all cases."

One not insignificant detail remains open. We speak of an event occurring with probability one, that is, an almost sure event. But which probability is meant here, since alongside the subjective probability we also have the risk-neutral probability Q? Fortunately, this question can be answered quickly, even without delving into the mathematical details: it makes no difference. The events that occur with probability one under the subjective probability P are precisely the same as those that occur with probability one under the risk-neutral probability Q. This detail is in fact included in the Fundamental Theorem of Asset Pricing; besides the pricing relation (1.9), the theorem entails this statement as well—a point we have not yet mentioned.

Taken together, these considerations show that the following assumption is both suitable and required.

Assumption 1.4 (Transversality) *If the lifespan of a firm goes to infinity ($T \to \infty$), then we have for every point in time t almost-everywhere*

[22] Convergence can be pointwise, almost-everywhere, in L^2 or in probability etc. (see Stoyanov (2014, Sect. 14) for an overview of different convergence concepts for random variables, their interrelations and numerous counterexamples).

[23] Notice that further an assumption is necessary that establishes a lower bound K of all cashflows at any time and any state. This lower bound can be arbitrary but need to be a finite number. We believe that this assumption is of technical nature that is not an important economic restriction. The following considerations are (in parts literally) taken from Kruschwitz and Löffler (2013).

$$\lim_{T\to\infty} \frac{\mathrm{E}_t^Q\left[\widetilde{V}_T\right]}{(1+r_f)^{T-t}} = 0\ . \tag{1.12}$$

Figure 1.7 illustrates the interplay between transversality and Fundamental Theorem on one hand and a unique firm value on the other hand. Transversality ensures that the Fundamental Theorem (1.9) and the valuation equation

$$\widetilde{V}_t = \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s\right]}{(1+r_f)^{s-t}} \tag{1.13}$$

are equivalent even in the case of infinitely lived firms. Those, who presume the Fundamental Theorem can utilize the valuation equation and vice versa if transversality holds. Exactly this is the meaning of Fig. 1.7.

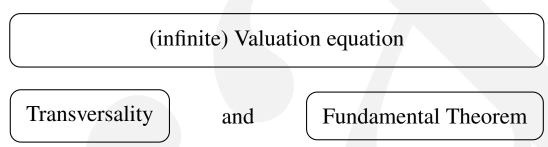

Fig. 1.7 Transversality is a sufficient and necessary condition for the infinite valuation equation.

Let us summarize. The transversality condition (1.12) is necessary if we want to value companies that stay on forever. Forgoing this assumption will lead to inconsistent results. Notwithstanding we want to emphasize that we do not believe in companies being perpetual forever. The main reason for dealing with these particular types of firms is that this assumption greatly simplifies our analysis.

1.4.6 Examples and Problems

1.4.6.1 The Finite Case (Continued)

Before proceeding, we want to take up our example from Sect. 1.3.3 and establish the value of the firm. We assume that the cost of capital of the firm is constant in time and amounts to $k = 20\%$. According to Thm. 1.1, it is obvious what market value the firm has at time $t = 0$,

$$\mathrm{V}_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_2\right]}{(1+k)^2} + \frac{\mathrm{E}\left[\widetilde{CF}^u\right]}{(1+k)^3}$$

$$= \frac{100}{1.2} + \frac{110}{1.2^2} + \frac{121}{1.2^3} \approx 229.75 \ .$$

Although the purpose of this calculation is perhaps not immediately evident here, we want to determine the market value of the firm for time $t = 1$ as well. This cannot be a number, because we cannot yet know today, if the outcome at time $t = 1$ will result in the condition up or down. Depending upon the condition, we are discounting different cash flows. In order to describe the details we must take into account at which node (state) we are looking at. In Sect. 1.3.1 we used the symbol ω for nodes. Any node is described by the movements that led to it; for example $\omega = uu$ is the node that came about after two up movements (in our example this corresponds to the cash flow 132). Let us denote with $\widetilde{CF}_2(u\omega)$ (or $\widetilde{CF}_2(d\omega)$) the cash flow at time $t = 2$ arriving at node ω given that the first movement was up (or down).

If we started with up, we get

$$\begin{aligned}\widetilde{V}_1(u) &= \frac{\mathrm{E}\left[\widetilde{CF}_2(u\omega)\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_3(u\omega)\right]}{(1+k)^2} \\ &= \frac{121}{1.2} + \frac{133.1}{1.2^2} \approx 193.26 \ ,\end{aligned}$$

while if we started with down

$$\begin{aligned}\widetilde{V}_1(d) &= \frac{\mathrm{E}\left[\widetilde{CF}_2(d\omega)\right]}{1+k} + \frac{\mathrm{E}\left[\widetilde{CF}_3(d\omega)\right]}{(1+k)^2} \\ &= \frac{99}{1.2} + \frac{108.9}{1.2^2} \approx 158.13\end{aligned}$$

is what we get. With that we get altogether

$$\widetilde{V}_1 \approx \begin{cases} 193.26, & \text{if the development in } t = 1 \text{ is up,} \\ 158.13, & \text{if the development in } t = 1 \text{ is down.} \end{cases}$$

Using the same technique the value of the unlevered firm at $t = 2$ is given by

$$\widetilde{V}_2 \approx \begin{cases} 121.00, & \text{if the development is up–up,} \\ 100.83, & \text{if the development is up–down or down-up,} \\ 80.67, & \text{if the development is down–down.} \end{cases}$$

1.4.6.2 The Infinite Case (Continued)

Let the cost of capital be $k = 20\%$. Then the value of the firm is using Rule 4 (remember $g = 0$)

$$V_0 = \sum_{t=1}^{\infty} \frac{\mathrm{E}_0\left[\widetilde{CF}_t\right]}{(1+k)^t} = \sum_{t=1}^{\infty} \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{(1+k)^t} = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{k^{E,u}} = \frac{100}{0.2} = 500 \; .$$

As in the finite case, we can use the Eq. (1.7) to calculate the firm's future values. We get

$$\widetilde{V}_t = \sum_{s=1}^{\infty} \frac{\mathrm{E}_t\left[\widetilde{CF}_s\right]}{(1+k)^s} = \sum_{s=1}^{\infty} \frac{\widetilde{CF}_t}{(1+k)^s} = \frac{\widetilde{CF}_t}{k} \; .$$

1.4.6.3 Problems

Problem 1.6 In the following problem, various types of returns are calculated. It turns out that these returns need not exhibit any meaningful relationship unless the stochastic structure of the cash flows is further restricted by additional assumptions.

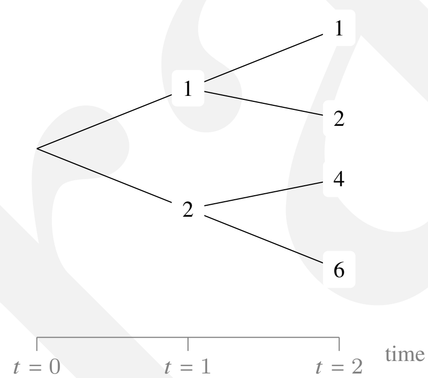

Fig. 1.8 Cash flows in Exercise 1.6.

The cash flows of a firm that exists until $T = 2$ are shown in Figure 1.8. The risk-neutral probabilities Q are each 50%, and the interest rate is $r_f = 0\%$ for simplicity. The market is free of arbitrage and the firm's values correspond to the (conditional) Q-probabilities.

a) Evaluate the firm values in $t = 0$ and $t = 1$.
b) Assume that the subjective probabilities are 40% for any up movement and 60% for any down smovement.
   Evaluate the following (conditionally) expected returns:

$$\frac{\mathrm{E}[\widetilde{CF}_2]}{\mathrm{E}[\widetilde{V}_1]} - 1, \quad \frac{\mathrm{E}_1[\widetilde{CF}_2]}{\widetilde{V}_1} - 1, \quad \frac{\mathrm{E}[\widetilde{V}_1 + \widetilde{CF}_1]}{V_0} - 1 \; .$$

Problem 1.7 Consider the return

$$\widetilde{r}_s := \frac{\widetilde{CF}_{s+1} + \widetilde{V}_{s+1}}{V_s} - 1 \ .$$

Assume that cost of capital (conditional expected returns) is deterministic. Show using only the rules of this chapter that for $s_1 > s_2 > t$ always

$$\mathrm{E}_t\left[\widetilde{r}_{s_1}\,\widetilde{r}_{s_2}\right] = \mathrm{E}_t\left[\widetilde{r}_{s_1}\right]\ \mathrm{E}_t\left[\widetilde{r}_{s_2}\right]\ .$$

This is also known as independence of returns.

Problem 1.8 Some textbooks consider firm values within a binomial tree while completely neglecting the associated cash flows. We demonstrate that such a model is internally inconsistent, since in this case firm values must necessarily be zero.

Suppose that a firm generates no cash flows at any point in time ($\widetilde{CF}_t = 0$).

a) Show that $\frac{\mathrm{E}_t^Q[\widetilde{V}_T]}{(1+r_f)^{T-t}} = \widetilde{V}_t$ holds.
b) Demonstrate that, under the transversality condition, one obtains $\widetilde{V}_t = 0$ for all t.

Problem 1.9 Consider the infinite example from Fig. 1.3. Assume that the cost of capital k are constant. Prove that

$$\widetilde{V}_{t+1} = \begin{cases} u\,\widetilde{V}_t & \text{if up,} \\ d\,\widetilde{V}_t & \text{if down.} \end{cases}$$

Show furthermore that (analogous to (1.7))

$$\mathrm{E}_s\left[\widetilde{V}_t\right] = \widetilde{V}_s\ .$$

Hint: Remember $g = 0$.

Problem 1.10 This problem will show that without transversality the value of a company will not be unique.

Assume that we have company values $\widetilde{V}_t$ and cash flows $\widetilde{CF}_t$ such that the valuation equation

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{V}_{t+1} + \widetilde{CF}_{t+1}]}{1 + r_f}$$

is satisfied. Transversality shall not hold. Show that the new "values" given by $\widetilde{V}_t^* =_{\mathrm{Def}} \widetilde{V}_t + C(1+r_f)^t$ with an arbitrary number $C \neq 0$ are also valid in the sense that these $\widetilde{V}_t^*$ satisfy the Fundamental Theorem (1.9) (still using the cash flows $\widetilde{CF}_t$!) as well.

Problem 1.11 [24] We consider a stochastic cash flow that, at time $t = 1$, can take only two possible values: u (up) or d (down). We assume that the both cash flows differ, $\widetilde{CF}_1(u) \neq \widetilde{CF}_1(d)$. We distinguish between the subjective probability P and the risk-neutral probability Q and assume they are not identical.

Suppose the subjective expected value of the cash flow is zero, $\mathrm{E}[\widetilde{CF}_1] = 0$. The prevailing intuition would then suggest that the firm's value is zero. In this particular setting, however, this conclusion is misleading:

[24] We are grateful to Denis M. Becker (Trondheim) for suggesting this problem.

a) Show that the risk-neutral expectation of the cash flow cannot be zero. The same holds for its value at $t = 0$.
b) Prove that the cost of capital of the cash flow is exactly -100% implying that DCF results in an indefinite term $V_0 = \frac{0}{0}$.

This shows that for cash flows with zero expected value, one should apply DCF methods with greater caution. One must be aware that the cost of capital must be such as to permit the application of the method in the first place.

Problem 1.12 Consider a one-period model. Using the pricing-kernel equation (1.11), it is straightforward to show that the following identity holds:

$$V_0 = \frac{\mathrm{E}_0\left[\widetilde{CF}_1\right]}{1 + r_f} + \mathrm{Cov}\left[m_1, \widetilde{CF}_1\right] \tag{1.14}$$

Show that this expression implies the security market line, given by

$$r_f = \mathrm{E}[r] + (r_f - \mathrm{E}[r_m])\beta \ ,$$

where $\beta := \frac{\mathrm{Cov}[r, r_m]}{\mathrm{Var}[r_m]}$ and $r := \frac{\widetilde{CF}_1}{\widetilde{V}_0} - 1$.

1.5 Further Reading

The concept of the conditional expectation goes back to the work of the Russian mathematician Kolmogorov from the 1930's and is found in every textbook on probability theory. The presentation given in the textbook by Williams (1991) is worth reading. Although it deals only with discrete-time, this textbook also gives a very good introduction to the theory of martingale measure. This also applies to Shreve (2004a). Those who want to read more about continuous-time models can turn to Karatzas and Shreve (1991), Musiela and Rutkowski (2009), Revuz and Yor (2005) or Shreve (2004b). Though all books are written for students majoring in mathematics.

The Fundamental Theorem of Asset Pricing was gradually recognized in several papers and is based on works from Beja (1971), Harrison and Kreps (1979) and Back and Pliska (1991). A proof can also be found in the textbook Musiela and Rutkowski (2009) and in Revuz and Yor (2005).

The definition and determination of cash flows is dealt with in-depth in every textbook on balance sheet analysis; Koller et al. (2015) is a good reference. The topic of the prognosis of future cash flows is unfortunately very often left out, Welch (2022, chapter 14) is a notable exception.

Rapp (2006) and Laitenberger (2006) discuss the question whether a suitable definition of cost of capital can be found that does not need the restriction of nonrandom returns.

Niemann (2004) analyzes stochastic tax rates and their impact on the neutrality of tax systems. An overview about national tax codes can be found in Koller et al. (2010, pp. 357–380). (The 2015 edition does not contain a chapter on national tax codes anymore.)

In Husmann et al. (2006) it is shown how several national tax codes can be implemented into our model.

Transversality is extensively discussed in Sethi (1996) as well as Kruschwitz and Löffler (2013). A lot of examples, counterexamples as well as proofs can be found there.

Asymmetric taxation with respect to valuation was discussed in Schaefer (1982), Dybvig and Ross (1986), Ross (1987) and Dermody and Rockafellar (1991).

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Chapter 2
Corporate Income Tax: APV, FTE, TCF, WACC

Abstract In this chapter, we examine taxes at the corporate level. These taxes imply that debt financing always increases firm value, and we aim to quantify the magnitude of this effect.
We first study the relationship between unlevered and levered firms and show why unlevered firms provide an appropriate reference point for DCF analysis. We discuss the stochastic properties of cash flows and argue that a martingale-like behavior is a sensible assumption. We then derive key implications of this assumption for firm valuation.
We then investigate how insolvency affects this value increase. To that end, we consider two typical insolvency triggers—illiquidity and over-indebtedness. We analyze what can be said about both triggers under a first approximation in which, upon default, all of the firm's assets are transferred to debtholders.
Subsequently, we introduce various financing policies and demonstrate how, in each case, the value differences between levered and unlevered firms can be determined.

Firm valuation is inherently difficult, which is why analysts rely on simplifying assumptions. Our first step is to consider an unlevered firm. As we will see, this benchmark already allows us to address several substantive economic questions. We then extend the analysis to levered firms.

2.1 Unlevered Firms

Anyone who values indebted firms must also be able to value their all-equity-financed counterparts. The two problems are logically interdependent.

Taken on its own, that claim is uninformative. Referencing an indebted firm without further specification leaves too much undefined: Are we looking at a highly or only moderately indebted firm? Is debt expected to increase, or is management planning to deleverage? By contrast, the benchmark of an all-equity-financed firm is transparent. By all-equity we mean a firm with no debt today and none at any future date. Such firms may be rare—perhaps nonexistent—in practice, but that empirical observation does not affect our argument. The point is definitional clarity: *all-equity* is precise, whereas *indebted* remains ambiguous without additional information.

Cost of Capital and Leverage A firm's cost of capital is driven by two forces: its business risk and its debt structure. All else equal, higher risk and a higher debt–equity ratio imply higher required expected returns. Linking this observation to the preceding discussion, the cost of capital for an all-equity firm is straightforward, whereas for an indebted firm it is a function of the level of debt. These statements are only valid so long as we keep all other influences on the cost of capital (particularly the business risk and the tax rate) constant.

The indebted firm's cost of capital is a prerequisite for valuing it. More precisely, we need the cost of capital of a firm that has two things in common with the firm to be valued: its business risk and its debt structure. In empirical work, one typically turns to capital-market data, identifies a comparable (peer) firm in the same—or at least a very similar—risk class, and estimates the expected return to its investors. Almost invariably, however, the peer's capital structure differs from that of the firm to be valued. If debt affects the cost of capital, the peer's figure cannot be transferred one-to-one. As noted above, indebted firms are not necessarily comparable even within the same risk class. And it is exactly here that the all-equity-financed firm comes into play as reference firm.

An indebted firm, this will become evident in a moment, is more valuable than an otherwise identical all-equity firm whenever interest is tax-deductible—holding the firm's operating (unlevered) cash flows fixed. Raising debt serves as leverage that can increase the firm's value. In this case one also speaks of a leverage effect caused by debt. Throughout, we use levered and indebted interchangeably, and likewise unlevered and equity-financed, in the strict all-equity sense defined above.

Unlevering and Relevering To determine the target firm's cost of capital, the peer's figure must be adjusted for the reasons just discussed. Academics, who are involved with the theory of valuation of firms, have to develop equations, which allow for the cost of capital of the (levered) peer firm to be mapped to that of an unlevered benchmark. If they are successful, then the equations can be used to infer the reference firm's cost of capital from the peer firm's cost of capital (unlevering), but also to infer the cost of capital of the firm to be valued from the reference firm's cost of capital (relevering).

And thus the circle is complete: whoever wants to value a levered firm must also be able to value an unlevered firm. Academics are then naturally required to live up to the expectations placed on them and must actually be in a position to develop the necessary adjustment equations. Should they fail to do so, the discounting of levered firms' cash flows with the appropriate cost of capital must simply be abandoned.

Notation In the first chapter of this book, we spoke of free cash flows and firm values without concerning ourselves with how the firms are financed. Now we concentrate on firms that are completely equity-financed. Accordingly, the relevant symbols carry an appropriate index. We use a superscript u for unlevered firms. We denote, for instance, the free cash flows after corporate income tax of such firms by $\widetilde{CF}_t^u$ and the firm values by $\widetilde{V}_t^u$. Note that, since only cash flows after taxes can be paid to the owners of the firm, $\widetilde{CF}_t^u$ denotes free cash flows after corporate income tax. Unlevered firms have a single group of financiers. For the returns that the owners expect, we use the symbol $\widetilde{k}_t^{E,u}$.

2.1.1 Valuation Equation and Discount Rates

In what follows, we assume that it is possible to derive the required adjustment equations, and we will attempt to do so here. Under this assumption, the all-equity firm's cost of capital can be treated as known. We assume that the analyst knows the unlevered firm's conditional expected free cash flows $\mathrm{E}_t\left[\widetilde{CF}_s^u\right]$ for time $s = t+1, \ldots, T$.

Definition 2.1 (Cost of capital of the unlevered firm) *Cost of capital $\widetilde{k}_t^{E,u}$ of an unlevered firm is the conditional expected return*

$$\widetilde{k}_t^{E,u} := \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u + \widetilde{V}_{t+1}^u\right]}{\widetilde{V}_t^u} - 1 .$$

The reader should note that we use cash flows after corporate income tax in our definition of the cost of capital. Therefore, $k_t^{E,u}$ are also after-tax variables. The question of how to infer any pre-tax cost of capital from these is not our concern, since we do not investigate how the value of a company changes with changes in the tax rate. Although possibly time-dependent, the tax rates are taken as fixed once and for all today. Nevertheless, anyone who tries to determine the pre-tax cost of capital cannot do so on the basis of our theory, since it says nothing about how the value of a firm changes with tax rates.

The valuation of the unlevered firm is straightforward under these conditions.

Theorem 2.1 (Market value of the unlevered firm) *If the unlevered firm's cost of capital $k_t^{E,u}$ is deterministic, then the value of the firm financed entirely with equity at time t equals*

$$\widetilde{V}_t^u = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{\left(1+k_t^{E,u}\right)\ldots\left(1+k_{s-1}^{E,u}\right)} .$$

We need not pursue the proof of this claim here. We have already addressed it in a more general form in Sect. 1.4.3 and need not bore the reader by repeating ourselves. The valuation equation, based on risk-neutral probabilities Q

$$\widetilde{V}_t^u = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^u\right]}{\left(1+r_f\right)^{s-t}}$$

can be derived analogously, without requiring a new proof.

We conclude this section by asking whether one should value the firm as a whole—taking all future cash flows together—or whether it is also possible to value individual cash flows separately. This issue raises the further question of how to define what we shall call discount rates.

Without further assumptions about the capital market, we cannot act as if a claim to this single cash flow were traded. If we did, a shareholder would have a claim to dividends, so to speak, but not to the security's price. Nevertheless, the question we ask is: what price should an investor pay at time $t < s$ for an isolated free cash flow $\widetilde{CF}^u_s$? If it were possible to sell a claim to a single cash flow at time t, at what price would that cash flow trade? This question becomes important when considering the tax advantages of debt: as we shall see, it is not merely a computational exercise but carries substantive meaning.

Although we have not set out the basic elements of arbitrage theory in detail, we may appeal to the Fundamental Theorem by an analogous argument. If we can value levered as well as unlevered firms with this principle, then it should also be possible to value the claim to an isolated cash flow. This cash flow is valued by taking the expectation with respect to the risk-neutral probability and then discounting at the risk-free rate,[1]

$$\frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_s\right]}{(1+r_f)^{s-t}} \ .$$

The above expression gives the value of the free cash flow $\widetilde{CF}^u_s$ at time t. It is immediately apparent that this valuation formula, albeit extremely elegant, is totally useless: we know next to nothing about the probability measure Q.

However, this expression provides a way to define the notion of a discount rate precisely. To that end, we make use of a few preliminary considerations. By the discount rate k'_t we mean the number that determines the price at time t of the cash flow $\widetilde{CF}^u_{t+1}$. According to our statements thus far, the discount rate k'_t shall serve as an instrument for valuing the single cash flow $\widetilde{CF}^u_{t+1}$, or, using subjective probabilities, we must have

$$\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{1+k'_t} = \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f} \ . \tag{2.1}$$

This formulation has the decisive advantage that, once k'_t is known, the value of an individual cash flow can be computed directly, since the subjective expectations (unlike Q) are observable.

Yet this consideration alone is not sufficient for our purposes. We do not want to use discount rates only to value cash flows that are a single period from the valuation date. For instance, if we value the cash flow $\widetilde{CF}^u_{t+2}$ at time t, we want to do so using two discount rates, k'_t and k'_{t+1}, in such a way that

[1] We have already noted that this procedure is permissible with regard to all conceivable payment claims, see Thm. 1.2.

$$\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+2}\right]}{(1+k'_t)(1+k'_{t+1})} = \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+2}\right]}{\left(1+r_f\right)^2} \tag{2.2}$$

is valid.

We emphasize that this is only one of many conceivable definitions. We could, for instance, define the discount rates as yields rather than in the way we have chosen. We refrain from doing so solely for practical reasons.

Definition 2.2 (Discount rates of the unlevered firm) *Discount rates of the unlevered firm's cash flow* $\widetilde{CF}^u_s$ *are real numbers* $k'_t, k'_{t+1}, \ldots$ *that satisfy*

$$\frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_s\right]}{(1+r_f)^{s-t}} = \frac{\mathrm{E}_t\left[\widetilde{CF}^u_s\right]}{\left(1+k'_t\right)\ldots\left(1+k'_{s-1}\right)} .$$

2.1.2 Martingale-Like Cash Flows

In Thm. 2.1, we derived a valuation equation for unlevered firms that the analyst can use only if she knows the unlevered firm's cost of equity. This condition can rarely be relied upon in practice. If the conditions required to use the theorem are not met, the valuation is anything but trivial.

Cash Flows of the Unlevered Firm One can only proceed in such a situation if the adjustment equation mentioned in the above section is available. The derivation of such an equation, however, is possible only if the appropriate assumptions are satisfied. In the following sections of this book, we will develop adjustment equations for challenging cases. Our results are based on a specific condition concerning the structure of the unlevered firm's free cash flows after taxes. To proceed, we first need to specify the stochastic structure of the unlevered cash flows. This step is important, as the assumptions we make here will reappear throughout the book. We deliberately begin with a structure that will soon prove inadequate. In doing so, we wish to emphasize at the outset why it is essential to carefully consider the stochastic properties of unlevered cash flows.

Independent Cash Flows? Let us assume for the moment that the unlevered cash flows at each point in time are independent of one another.[2] Such an assumption could be motivated if the company's past cash flows contained no information about its future cash flows. This, however, is hardly convincing: every company pursues a business strategy, and the successes or failures of that strategy become increasingly evident over

[2] This case has already been investigated by Froot and Obstfeld (1991), who assumed that the cash flows follow a white noise process.

time. Under such circumstances, independence cannot be maintained. As noted, our aim here is not to defend this assumption as realistic, but to illustrate why it is best avoided.

If, however, this condition were satisfied, the following would hold. As we have already noted, conditional expectations need not be numbers; they may themselves be random variables. But with independent cash flows, the situation is different: for every t, the conditional expectation $\mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u]$ is a definite number rather than a random variable.[3]

By applying the law of iterated expectations (Rule 4, Iterated Expectations), we can derive an even stronger result. It follows for the conditional expected value at $t < s-1$ that

$$\mathrm{E}_t^Q[\widetilde{CF}_s^u] = \mathrm{E}_t^Q\left[\mathrm{E}_{s-1}^Q[\widetilde{CF}_s^u]\right]$$

Since the inner expectation on the right is a number, the conditional expectation on the left must also be a number (Rule 3, Certainty).

Looking at the valuation Eq. (1.10),

$$V_t^u = \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t^Q[\widetilde{CF}_s^u]}{(1+r_f)^{s-t}},$$

we see that every summand is a number. Although cash flows may fluctuate considerably due to uncertainty, the firm's value does not (we even omitted the tilde on the left-hand side for that reason).

If independence is imposed as the stochastic structure of cash flows, the implications for firm value turn out to be implausible. We do not observe cash flows or dividends to be stable while the associated firm values or stock prices show no volatility; if anything, the opposite is true. This should not be surprising: an implausible assumption has led to an implausible conclusion. Before discarding this assumption, however, it is worth pursuing the example a bit further, as it still yields useful insights.

First, anyone who applies our definition of the cost of capital will find that it does not suffer from the shortcomings discussed at length in Sect. 1.4.2. With independent cash flows, the cost of capital is no longer a random variable but is deterministic. Accordingly, no additional assumption about the cost of capital is required to determine the firm's value by discounting.

Second, let us add another property to the cash flows to reflect further on the relationship between the concepts introduced here. In the problems,[4] we require that the cash flows be not only independent but also identically distributed—commonly abbreviated as iid. In that problem we show that this implies that firm values are deterministic and identical at every point in time. This result should not come as a surprise: if all future cash flows are identically distributed random variables, there is no

[3] This result cannot be derived from the rules for conditional expectations set out in Sect. 1.3.2. Strictly speaking, it would have to be stated as an additional rule. We refrain from doing so, since this case arises only in the present section of this book.

This conclusion holds for any probability measure, not only Q. It follows that the same applies under the subjective probability P, so that $\mathrm{E}_t[\widetilde{CF}_{t+1}^u]$ must likewise be a number.

[4] See Prob. 2.1.

reason why firm values (as claims on those cash flows) should differ. What follows from this observation?

As it turns out, the cost of capital is constant because every element in its definition is constant. Let us denote this value by k. We can then take the analysis one step further. In a moment, we will address a question that may seem surprising at first glance.

In Prob. 2.1, we verify that the discount rate k'_t can be determined explicitly in the case of independent cash flows. We then demonstrate that this rate must differ from the firm's overall cost of capital, k. In other words, the cost of capital for the entire firm and the discount rates for individual cash flows should not be assumed to coincide. With this insight, we conclude our illustrative (though admittedly peculiar) example and turn to a stochastic structure for cash flows that promises to yield more meaningful results.

Martingale-Like Cash Flows! We now assume that the cash flows follow a process we call martingale-type.[5] The reader may rest assured that, although this assumption is mathematically less straightforward than the preceding case, it allows us to derive economically meaningful results.

What exactly do we mean by a martingale-like process? The idea is best conveyed by contrasting it with our earlier example. There, we established that past cash flows provided no information about the next period's cash flow. This changes now. In martingale-like processes, the cash flow observed at a given time and state is the best available predictor of future cash flows. If cash flows at a given time and state are rising, one expects them to continue rising; if they are declining, one expects future cash flows to remain low. We state this idea precisely below. This assumption aligns with the view of a firm pursuing a consistent strategy and seeing the results reflected in its cash flows.

Assumption 2.1 (Martingale-like cash flows) *There is a real number $g > -1$ such that*

$$\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] = (1+g)\,\widetilde{CF}^u_t$$

is valid for the unlevered firm's cash flows after taxes.

In earlier editions, we allowed the growth rate g to vary over time, and that generalization is certainly possible. In our view, however, this approach has two major drawbacks. First, it is virtually never encountered in practice, where only a constant g (if at all) is used. Second, it renders the theoretical statements unnecessarily cumbersome, since one must replace the simple power $(1+g)^t$ with the product of all past growth factors.

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[5] In earlier editions, we used the term "(weak) autoregressive cash flows". This terminology, however, is misleading. An autoregressive process includes a constant term; that is, there exists a number $\boldsymbol{C} \neq 0$ such that $\mathrm{E}_t[\widetilde{CF}^u_{t+1}] = \boldsymbol{C} + (1+g), \widetilde{CF}^u_t$. In our definition below, we do not assume the existence of such a $\boldsymbol{C}$. Hence, we have chosen a mathematically more precise description.

Martingales were introduced into probability theory by the American mathematician Joseph L. Doob. Strictly speaking, under the definition below only the case $g = 0$ is called a martingale. If $g < 0$, it is called a supermartingale; if $g > 0$, a submartingale. We subsume all three cases under the term "martingale-like," since for our purposes the sign of the growth rate g does not matter.

For readability, we therefore refrain from this generalization in this edition. Readers interested in the more general case will find it straightforward to extend the results.

We must assume that g is greater than -100% to prevent cash flows from having alternating signs, which would be unrealistic. It is not necessary to assume that g is positive; thus shrinking cash flows are not excluded under Assump. 2.1.

Justification of Martingale-Like Cash Flows Every economic theory is based on assumptions. The many jokes told about economists stem from the fact that we sometimes employ unrealistic or odd assumptions. Economists who wish to be taken seriously must face the question of whether their assumptions are justifiable. Can we, in good conscience, claim that Assump. 2.1 holds for a firm's free cash flows? Isn't this assumption perhaps "far-fetched"?

From our experience, practitioners generally do not concern themselves with the distributional laws of future stochastic cash flows. Instead, they limit themselves to estimating the expectations of these future cash flows. It can be shown that, for any sequence of expectations, there exists a state space with upward and downward movements like the one in Fig. 1.2. Assump. 2.1 is therefore satisfied in this model.[6] This means simply that an analyst working with estimated cash-flow expectations can proceed on the basis that—so to speak, behind the scenes—there is always a state space consistent with Assump. 2.1. All in all, we regard the martingale-like cash-flow assumption as practically acceptable.

Moreover, there is another point to consider. Following Milton Friedman, our concern should be less with the realism of our assumptions and more with the results of our theory. We will return to this issue once we can present the first results of our theory (see Sect. 2.1.4).

At this point one may ask why exactly the *unlevered* firm's cash flows should be martingale-like. It is clear that Assump. 2.1 appears arbitrary, and there is no point in concealing this. Could we not just as well replace it with the condition that the free cash flows of a *levered* firm are martingale-like? Could this levered firm be chosen arbitrarily, or would it need a particular financing policy? We must answer these questions clearly: the valuation equations we develop below would, in any case, yield different results. Moreover, the following observation is important: even if a levered firm has martingale-like cash flows, it does not necessarily follow that the free cash flows of a firm with a different financing policy share this property.[7]

Uncorrelated (Additive) Increments To understand our assumption, we first consider the "increments" of the cash-flow process. Formally, they are defined by the equation[8]

$$\widetilde{CF}^u_{t+1} = (1+g)\,\widetilde{CF}^u_t + \varepsilon_{t+1}\ . \tag{2.3}$$

[6] The proof requires rather involved calculations, which we omit here. Anyone who takes the trouble to revisit our example in Fig. 1.2 will find that the cash flows in that example are martingale-like. See also Prob. 1.4, which is devoted to this calculation.

[7] Prob. 2.12 is devoted to this point.

[8] For the random variable ε, we omit the tilde notation.

g is a deterministic quantity known at $t = 0$. What does our assumption about martingale-like cash flows imply for the increments ε_{t+1}? It implies that these increments have mean zero and are uncorrelated with one another.

It is not immediately obvious that this assumption implies uncorrelated noise terms. To prove this, we perform a short calculation, again using the rules for conditional expectations. First, we show that the noise terms have mean zero,

$$\begin{aligned}
\mathrm{E}[\varepsilon_{t+1}] &= \mathrm{E}_0[\varepsilon_{t+1}] && \text{Rule 1}\\
&= \mathrm{E}_0[\mathrm{E}_t[\varepsilon_{t+1}]] && \text{Rule 4}\\
&= \mathrm{E}_0\left[\mathrm{E}_t\left[\widetilde{CF}^u_{t+1} - (1+g)\widetilde{CF}^u_t\right]\right] && \text{by (2.3)}\\
&= \mathrm{E}_0\left[\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] - (1+g)\,\mathrm{E}_t\left[\widetilde{CF}^u_t\right]\right] && \text{Rule 2}\\
&= \mathrm{E}_0\left[\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] - (1+g)\widetilde{CF}^u_t\right] && \text{Rule 5}\\
&= \mathrm{E}_0[0] && \text{Assump. 2.1}\\
&= 0\ .
\end{aligned}$$

Now we turn to the proof that the noise terms are uncorrelated. We consider two times $s < t$ and must show that the covariance $\mathrm{Cov}[\varepsilon_s, \varepsilon_t]$ is zero. We already know that the noise terms have mean zero. Thus, the covariance reduces to

$$\begin{aligned}
\mathrm{Cov}[\varepsilon_s, \varepsilon_t] &= \mathrm{E}[\varepsilon_s\varepsilon_t] - \mathrm{E}[\varepsilon_s]\,\mathrm{E}[\varepsilon_t]\\
&= \mathrm{E}[\varepsilon_s\varepsilon_t]\ .
\end{aligned}$$

It follows from the rules and from the definition of the noise term that

$$\begin{aligned}
\mathrm{Cov}[\varepsilon_s, \varepsilon_t] &= \mathrm{E}[\varepsilon_s\varepsilon_t]\\
&= \mathrm{E}_0[\varepsilon_s\varepsilon_t] && \text{Rule 1}\\
&= \mathrm{E}_0[\mathrm{E}_s[\varepsilon_s\varepsilon_t]] && \text{Rule 4}\\
&= \mathrm{E}_0[\varepsilon_s\,\mathrm{E}_s[\varepsilon_t]]\ . && \text{Rule 5}
\end{aligned}$$

We now focus on the conditional expectation appearing in the last equation and, using the rules together with the martingale-like property of cash flows, obtain

$$\begin{aligned}
\mathrm{E}_s[\varepsilon_t] &= \mathrm{E}_s\left[\widetilde{CF}^u_t - (1+g)\widetilde{CF}^u_{t-1}\right] && \text{by (2.3)}\\
&= \mathrm{E}_s\left[\mathrm{E}_{t-1}\left[\widetilde{CF}^u_t - (1+g)\widetilde{CF}^u_{t-1}\right]\right] && \text{Rule 4}\\
&= \mathrm{E}_s\left[\mathrm{E}_{t-1}\left[\widetilde{CF}^u_t\right] - (1+g)\widetilde{CF}^u_{t-1}\right] && \text{Rule 2, 5}\\
&= 0\ . && \text{Assump. 2.1}
\end{aligned}$$

The covariance disappears, which is exactly what we wanted to show.

Independent (Additive) Increments Until now, we have proven that the vanishing expectation of the noise terms ε_{t+1} as well as their uncorrelatedness results from the Assump. 2.1. You may get the impression that the reverse is true as well. This is not the case. For the sake of completeness, we have to ascertain that it is the condition

$$\mathrm{E}_t\left[\varepsilon_{t+1}\right] = \mathrm{E}\left[\varepsilon_{t+1}\right] (= 0) ,$$

which is logically sufficient for martingale-like cash flows.

To understand this condition, note that many financial models are based on the assumption that a firm's free cash flows follow a random walk. What does that mean? It means the cash flows have increments that are independent and identically distributed (iid). The noise terms ε_{t+1} have mean zero, are identically distributed, and are mutually independent.

This is a strong assumption—much stronger than the one we made. The independence of the noise terms implies, for instance, that the increment of the cash flow in period t is unrelated to the cash flows from periods 1 to $t-1$. Even if cash flows have shown continuous growth in recent years, this assumption means that such growth provides no indication whatsoever that it will continue in period t. In other words, nothing about the development of cash flows from periods 1 to $t-1$ allows us to infer anything about period t. Uncorrelatedness, by contrast, does not impose such a strong restriction.

While uncorrelatedness excludes only a linear relationship, independence rules out any causal relationship whatsoever. In our notation, independence is equivalent to

$$\mathrm{E}_t\left[f(\varepsilon_{t+1})\right] = \mathrm{E}\left[f(\varepsilon_{t+1})\right]$$

for any function $f(x)$.[9] Compare this with the equation above, where f need only be linear. If the equation does not hold for any linear f, some nonlinear functional form may still satisfy it. Independent random variables always have zero correlation, but (except in the Gaussian case) uncorrelated random variables need not be independent. This underscores that our martingale-like cash-flow assumption is a weaker condition. We are not assuming a random walk for the cash flows; rather, we work with the less demanding assumption of uncorrelated growth.

Multiplicative versus Additive Increments Understanding martingale-like cash flows is not easy. In the previous section, we examined additive increments of cash flows to find an appropriate interpretation. But the additive context is by no means compelling; we could just as well use a multiplicative link. Below, we briefly discuss this and show that the result does not change. Thus, whether we take the additive or multiplicative route is a matter of taste. A multiplicative relation is defined by

$$\widetilde{CF}^{u}_{t+1} = (1+g)\,\widetilde{CF}^{u}_{t}\,(1+\varepsilon_{t+1}) \ . \tag{2.4}$$

First, we will check which properties of the error terms ε_t guarantee that the cash flows turn out to be martingale-like once again. By inserting Assump. 2.1 in the definition we get

[9] See for example Ingersoll Jr. (1987, p. 15).

$$\begin{aligned}(1+g)\widetilde{CF}^u_t &= \mathrm{E}_t[\widetilde{CF}^u_{t+1}] && \text{Assump. 2.1}\\ &= (1+g)\,\widetilde{CF}^u_t\ \mathrm{E}_t[1+\varepsilon_{t+1}] && \text{Rule 5}\\ &= (1+g)\,\widetilde{CF}^u_t\ (\mathrm{E}_t[1]+\mathrm{E}_t[\varepsilon_{t+1}]) && \text{Rule 2}\\ &= (1+g)\,\widetilde{CF}^u_t\ (1+\mathrm{E}_t[\varepsilon_{t+1}]) && \text{Rule 3}\ .\end{aligned}$$

This means that

$$\mathrm{E}_t[\varepsilon_{t+1}] = 0$$

is sufficient and necessary for being martingale-like. Differences between additive and multiplicative increments only play a role when the distributions of the error terms come into play. However, these differences are not important in the following. The result can also be seen by taking the logarithm on both sides of Eq. (2.4),

$$\log\left(\widetilde{CF}^u_{t+1}\right) = \log\left((1+g)\widetilde{CF}^u_t\right) + \log(1+\varepsilon_{t+1})\ .$$

In a multiplicative model the logarithms of the cash flows follow an additive model with $\log(1+\varepsilon_{t+1})$ representing the error terms. Based on these considerations we will restrict ourselves on additive error-terms from now on.

2.1.3 Gordon-Shapiro Formula, Discount Rates Again, and a First Look at Insolvency

We now turn to a first result for the case of martingale-like cash flows. It can be shown that the unlevered firm's value is a multiple of its free cash flow. Put differently, the unlevered firm has a deterministic dividend–price ratio.

So far, we have assumed that the cost of capital $k^{E,u}_t$ may vary over time. Of course, in such a setting one can also establish a general result, as we did in earlier editions. For didactic reasons, however, we believe it is more straightforward to confine ourselves here to the case of a constant cost of capital $k^{E,u}$; in what follows, we will restrict our analysis to this special case, well known as the Williams/Gordon-Shapiro formula.

Theorem 2.2 (Williams/Gordon-Shapiro formula) *Let the cost of capital be deterministic, time-independent and the cash flows are martingale-like. If $k^{E,u} > g$, then for the value of the unlevered firm*

$$\widetilde{V}^u_t = \frac{1+g}{k^{E,u}-g}\,\widetilde{CF}^u_t$$

holds, otherwise the firm value is infinite. The factor $\frac{1+g}{k^{E,u}-g}$ is also called price-dividend ratio.

We have relegated the proof for this theorem to the appendix.

The last proposition shows that the expected capital gains of the unlevered firm rate is deterministic

$$\text{gain}_t = \frac{\mathrm{E}_t\left[\widetilde{V}^u_{t+1}\right] - \widetilde{V}^u_t}{\widetilde{V}^u_t} = \frac{\frac{1+g}{k^{E,u}-g}\ \mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{\frac{1+g}{k^{E,u}-g}\ \widetilde{CF}^u_t} - 1 = g$$

and equal to the growth rate of the cash flows. Also, the expected dividend rate

$$\begin{aligned}\text{div}_t &= \frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{\widetilde{V}^u_t}\\ &= \frac{(1+g)\,\widetilde{CF}^u_t}{\widetilde{V}^u_t} = (1+g)\frac{k^{E,u}-g}{1+g} = k^{E,u} - g\end{aligned}$$

is equal to the difference between the cost of capital and the growth rate.

Discount Rates We have already made it clear in the introduction that, in the case of certainty, returns, not yields, are the appropriate way to determine the value of cash flows. We now consider the relationship between returns and discount rates. This is a second implication of cash flows being martingale-like. If this is the case, then there is another way to value them that is of interest to us: using the discount rates introduced in Def. 2.2.

Put differently, the cost of capital is not merely an appropriate discount rate; it has the further advantage of being independent of the time at which the cash flow occurs (the same $k^{E,u}$ applies for all s). This property will prove very helpful later.

Theorem 2.3 (Cost of capital is a discount rate) *If the cost of capital is deterministic and the unlevered firm's cash flows are martingale-like, then the following holds for all t: $s > t$*

$$\frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_s\right]}{\left(1+r_f\right)^{s-t}} = \frac{\mathrm{E}_t\left[\widetilde{CF}^u_s\right]}{\left(1+k^{E,u}_t\right)\ldots\left(1+k^{E,u}_{s-1}\right)}\ .$$

This is the same as to say that cost of capital is indeed a discount rate.

That this theorem follows from the assumptions made here can hardly be so easily recognized. Since the proof requires a fair amount of space and perhaps not even be of interest to every reader, we have relegated it to an appendix.[10]

Critical readers could suspect that this theorem has to do with a simple application of the definition of the cost of capital. This would most definitely be a wrong conclusion and in order to make it more understandable, we would like to go into it in more detail. Just equating Eq. (1.10) with Thm. 2.1 leads us in terms of unlevered firms to the result

$$\widetilde{V}_t^u = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^u\right]}{\left(1+r_f\right)^{s-t}} = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{\left(1+k_t^{E,u}\right)\dots\left(1+k_{s-1}^{E,u}\right)} .$$

The preceding equation is of little surprise, as it only claims the equivalence of two different ways of calculation: either the risk-neutral probability and the risk-free interest rate are used, or the analyst applies the subjective probability and the correspondingly defined cost of capital. Since the cost of capital is now so defined that both expressions give identical values, there is no reason to worry about coming up with equal firm values.

Of surprise, however, is the declaration that not only do the sums agree in the last equation, but the summands as well. This is everything but obvious, as the simple example

$$4 + 6 = 3 + 7 \qquad \text{but } 4 \neq 3 \text{ and } 6 \neq 7$$

shows. The reader should keep both statements (i.e., the identity of the sums as well as the identity of the summands) clearly separate. Our statement is everything but self-evident and is thoroughly in need of a proof.

A First Look at Insolvency Until now, we have deliberately avoided discussing the possibility that the firm being valued might fail. In projecting future cash flows and in determining future financing and investment policies, all conceivable developments are taken into account. As the firm expands and enters into payment obligations—toward suppliers, employees, or tax authorities, for example—the likelihood of financial distress increases.

Any careful analysis must begin by drawing several distinctions. In financial practice, the relevant terms are often used inconsistently, and even some textbooks do not set a clear or reliable example.[11] The following clarifies how we shall use and distinguish these terms:

FINANCIAL DISTRESS This term is loosely defined. It usually refers to a firm experiencing financial difficulties. However, a notion this vague has no place in a formal model. For modeling purposes, we require situations that can be clearly identified and precisely described.

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[10] In the theorem, the cost of capital carries a time index, whereas the risk-free interest rate does not. As noted, we assume a flat term structure; however, the theorem can readily be generalized to a non-flat term structure as well.

[11] Earlier editions of this work, unfortunately, provide a regrettable illustration of this.

Insolvency This term refers to a situation that can be precisely specified and verified. Supplier and service contracts often address this explicitly: once insolvency occurs, the agreed contractual consequences are set in motion.
Examples include an invoice that cannot be paid when due or a balance sheet showing negative equity. Whether such circumstances have consequences, and what form they take, depends on the specific provisions of the contract.

Default Insolvency is a condition; default is an event. Once insolvency has been established, the question arises as to what consequences should follow. In this context, a default means a breach of contractual obligations.
We will not analyze default in full detail here, for a simple reason: our framework can determine, for instance, when a payment is missed, but what happens thereafter depends on the contract. The firm may be placed under new management without a change of ownership (German law refers to this as "Eigenverwaltung"), or it may be liquidated entirely. Since the detailed provisions of a loan contract lie outside our model, we do not specify a full default procedure. We will, however, examine selected default arrangements where necessary to illustrate their implications for valuation.

Bankruptcy Bankruptcy refers to the legal procedure that begins once insolvency has been established and a default process has been initiated. Such legal proceedings fall outside the scope of the present analysis.

In what follows, we focus on questions of insolvency. From this point onward, the term *insolvency* will be used strictly in the sense just defined. Occasionally, for stylistic variety, we may also use the expression *go bankrupt*. Both terms refer to the same phenomenon.

When we discuss insolvency, we are talking about a condition that occurs. This naturally raises the question of the extent to which this condition is related to the circumstances that influence cash flows and company values. There are two ways of dealing with this question.

- In so-called reduced-form models, insolvency is driven by conditions that are not modeled as a direct consequence of the firm's cash flows or value. Instead, default is introduced as an external event (relative to these variables). An insolvency event, so to speak, comes out of nowhere.
- The picture is different in structural models: by contrast, structural models tie insolvency triggers directly to the economic conditions that shape cash flows and firm value.

Since this book places great emphasis on modeling the stochastic structure of cash flows, we will focus exclusively on structural models. In particular, insolvency triggers will be tied to specific cash-flow events.

Two potential triggers of insolvency will be central to our later analysis: the inability to pay and the loss of equity. Many bankruptcy laws around the world recognize these two as insolvency triggers.[12]

[12] The United States is an exception, as only the balance-sheet test (§ 101(32)(A) Bankruptcy Code) is explicitly defined there. German law goes further, adding a third trigger, "drohende Zahlungsunfähigkeit" (§ 18 Insolvenzordnung), which applies when future inability to pay becomes "more likely than not."

A company is over-indebted when its net assets are negative. Net assets can be assessed using either book values or market values. Since we take a financial-theory perspective, we focus exclusively on market values and set accounting issues aside.

Unlevered Firms and Insolvency Unlevered firms may become insolvent if the claims of tax authorities, employees, and other claimants cannot be satisfied. Formal insolvency proceedings are regulated differently from one jurisdiction to another. We have already mentioned that illiquidity and over-indebtedness are typical insolvency triggers: a company gets illiquid if its net cashflows are negative. A company is over-indebted if the value of equity is negative.

We define an unlevered firm as illiquid if the owners' net cash flow at time t is negative, i.e., $\widetilde{CF}_t^u(\omega) < 0$ (equivalently, the owners are unable or unwilling to meet their funding obligations at t).

Let us turn to over-indebtedness as the second insolvency trigger. How could this term be interpreted when we look at an unlevered firm? We characterize such a situation by $\widetilde{V}_t^u(\omega) < 0$ (equivalently, the owner's equity becomes negative). If this condition is fulfilled, from the owner's point of view continuing the business would be out of the question.

The following proposition shows that, for unlevered firms, over-indebtedness and illiquidity are equivalent: the two insolvency triggers coincide, so distinguishing between them would add little. For levered firms, however, the situation will become much more complicated.

Theorem 2.4 (Insolvency of the unlevered firm) *Consider an unlevered firm with martingale-like cashflows and a constant cost of capital. If the company is illiquid at date t in some state, it is also be over-indebted at the same date in the same state.*

This immediately follows from Thm. 2.2. □

Notice that the market value of equity for corporations cannot be negative, since they have limited liability. The market value of non-corporations, say partnerships, can be negative though. Our theorem mainly illustrates that one must check whether the valuation model's structural implications fit the real-world setting one intends to discuss.

We now have the basic elements of our theory of discounted cash flow. In the sections that follow, we show what can be done with these elements. Before proceeding, we return to a question raised earlier but set aside when formulating our assumption of a deterministic cost of capital: How realistic are the consequences that follow from this assumption?

2.1.4 Excursus: Excess Volatility and Stochastic Cost of Capital

Readers of this book are primarily concerned with how firms can be valued. Yet in practical application it quickly becomes clear that professionals face considerable obstacles. More often than not, they are forced to strike compromises in order to arrive at results that can at least be regarded as defensible. One of the most vexing difficulties arises already at the stage of gathering basic information: even seemingly straightforward elements such as cash flow or the cost of capital frequently prove inaccessible or insufficiently reliable. Under such conditions, the question of whether the valuation equations rest on impeccable theoretical foundations often loses importance, as the priority shifts to what can actually be implemented in practice.

Scientific methods differ fundamentally from practical approaches. Accepting compromises under the pressure of time is out of question; the focus is instead on the internal consistency and verifiability of results. The assumptions underlying the Gordon–Shapiro Theorem 2.2 have been presented here in detail and comprehensively. What matters is that applying this equation—as well as the more general valuation Eq. (1.1)—necessarily entails treating a firm's (inherently uncertain) cost of capital as if it were deterministic. [13]

When formulating this assumption, it has been described as "heroic." We now want to examine the extent to which such an assumption can be regarded as justified. In practice, such a question arises only if several alternative assumptions for valuation are available—a situation which, to the best of current knowledge, does not exist. For this reason, the ensuing discussion is marked as a digression. The first step must be to clarify the criterion by which the assumption is to be assessed.

Since Milton Friedman, the realism of assumptions has not been the standard for judging a theory; rather, it is the explanatory and predictive power of the results derived from them that serves as the decisive criterion:

> "The relevant question to ask about the 'assumptions' of a theory is not whether they are descriptively 'realistic,' for they never are, but whether they are sufficiently good approximations for the purpose in hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions," Friedman (1953, p. 15).

The same applies, in equal measure, to DCF.

A glance at the Gordon–Shapiro Theorem 2.2 makes it clear that the value of a firm and its cash flow at a given point in time differ only by a constant. If firm values and cash flows are regarded as random variables, there is an immediate and close dependence between their stochastic structures. Assuming a particular distribution for the firm's future value therefore entails corresponding requirements for the distribution of its future cash flow. For example, if the firm's value is assumed to be lognormally distributed, then (according to the Gordon–Shapiro equation) the cash flow must likewise follow a lognormal distribution; otherwise, the equation could not be satisfied consistently.

Such a relationship can be tested empirically if one assumes that future firm values and cash flows can be inferred from values observed in the past. In this context, it is natural to focus on publicly listed companies, simply because abundant data are

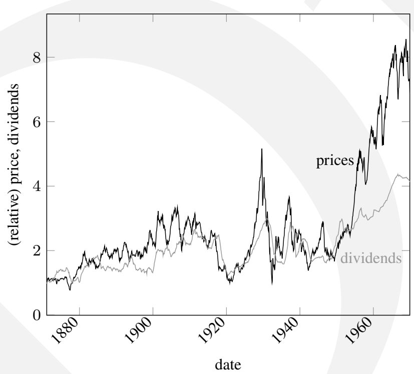
[13] See Assump. 1.2.

available for them. At the same time, one must keep in mind that the Gordon-Shapiro Theorem 2.2 speaks to a specific firm (which must moreover be fully equity-financed); for that reason, many empirical studies have examined stock market indices rather than individual companies.

Stock market reporting routinely emphasizes that the prices of individual stocks, entire portfolios, and indices fluctuate quite substantially. By contrast, comparable statements about dividends (which in any case occur only at longer intervals) are relatively rare. An examination of longer time series makes it clear that this is hardly surprising: stock prices typically display much greater volatility than dividends. Such an observation, however, pertains only to historical developments and does not in itself allow for direct inferences about future relationships—even if the data suggest a seeming regularity that might lead one to expect a degree of persistence.

In addition, any comparison of stock prices and dividends must take into account not only the differing observation intervals but also the substantial disparity in magnitude between the two variables. A systematic analysis of this question was first undertaken in Shiller (1981). For this purpose, Shiller constructed an index comparable to the S&P 500, but tracing developments on a monthly basis back to 1870. His index incorporates both stock prices and the dividends paid in each respective month. The results of this investigation will be presented graphically below.

For the analysis, the period from 1870 to the present is divided into two subintervals. Fig. 2.1 depicts the evolution of dividends and stock prices. In both subperiods it is evident that the volatility of stock prices clearly exceeds that of dividends. This finding thus empirically confirms the earlier supposition.[14] The title of Shiller's article captures this phenomenon with precision: he speaks of "excess volatility." The results of this investigation have been repeatedly tested and confirmed by Shiller himself as well as by numerous other authors. In current research, its validity is hardly called into question.



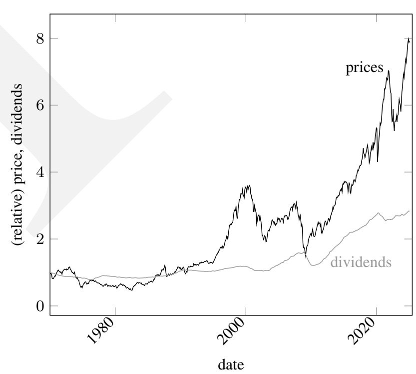


Fig. 2.1 US stock prices and dividends from www.shillerdata.com, left from 1870-1969 (normalized to 1 in 1870) and right from 1970-2025 (normalized to 1 in 1970).

[14] Purely graphical evidence can be reinforced by quantitative evaluation. The empirical coefficient of variation of prices, discussed in more detail below, amounts to 1.28, whereas the coefficient of variation of dividends is substantially lower at 0.693.

Linking the empirical evidence presented above with the Gordon–Shapiro Theorem 2.2 is far from straightforward. Since this equation does not provide graphical evidence but only numerical measures, it is necessary to employ an appropriate statistic. As already noted, dividends and stock prices differ greatly in magnitude, which makes a direct comparison of their volatility difficult. Against this background, the coefficient of variation suggests itself as a suitable measure, since it relates the standard deviation to the expected value. For a random variable $\widetilde{X}$, it is defined as

$$\frac{\operatorname{Std}\left[\widetilde{X}\right]}{\operatorname{E}\left[\widetilde{X}\right]} \ .$$

When this coefficient of variation is calculated for firm value and cash flows, a surprising result emerges:

$$\frac{\operatorname{Std}\left[\widetilde{V}_t^u\right]}{\operatorname{E}\left[\widetilde{V}_t^u\right]} = \frac{\operatorname{Std}\left[\frac{1+g}{k^{E,u}-g}, \widetilde{CF}_t^u\right]}{\operatorname{E}\left[\frac{1+g}{k^{E,u}-g}, \widetilde{CF}_t^u\right]} = \frac{\operatorname{Std}\left[\widetilde{CF}_t^u\right]}{\operatorname{E}\left[\widetilde{CF}_t^u\right]} \ ,$$

This implies that, according to the Gordon–Shapiro setup, both quantities should exhibit the same degree of relative variability. What we observed in the Fig. 2.1 above does not accord with what our theory posits.

At first glance, the situation seems clear: a problematic assumption has been formulated. This leads to consequences that cannot be empirically verified. Accordingly, one would have to develop an alternative theory. Yet the matter does not prove to be quite so straightforward.

First, in the Gordon–Shapiro theorem, we did not only assume constant cost of capital. The martingale-like structure of the cash flows also played an important role in the proof.[15] Unlike in the case of cost of capital, however, we took care to substantiate this assumption about cash flows with sufficient economic justification. We also demonstrated what happens when alternative assumptions are adopted. For this reason, in what follows we shall not focus on the assumption concerning cash flows, but rather on what we have postulated with respect to cost of capital.

It should then be noted that the Gordon-Shapiro Theorem not only assumed deterministic cost of capital but also their constancy over time. Against this background, the question arises whether the observed excess volatility is primarily due to the assumption of time-invariant cost of capital, without the absence of uncertainty in the parameter $\widetilde{k}_t$ necessarily being decisive. In other words: could a cost of capital that is deterministic yet time-varying already suffice to explain the discrepancy, such that the lack of a stochastic component in $\widetilde{k}_t$ is not crucial?

Two considerations may be put forward in this regard. First, in earlier editions, the cost of capital was assumed to be non-stochastic, though possibly time-varying. Nevertheless, it was still possible to derive a statement comparable to the Gordon-Shapiro Theorem. The calculations carried out in connection with the two coefficients of variation remained unchanged: the coefficient of cash flows would still be identical to that of firm values—

[15] See Sect. 2.1.2.

even in the case of time-varying cost of capital. At its core, this relationship was already recognized by Shiller, who, on the basis of his empirical investigations, concluded:

> "I have shown, however, that the movements in expected real interest rates that would justify the variability in stock prices are very large—much larger than the movements in nominal interest rates over the sample period," Shiller (1981, p. 434).

In more recent literature, numerous attempts have been made to account for time variation in cost of capital; in this context, the work of Campbell and Shiller (1988), with the approximation equation named after the authors, is regarded as the standard reference. Still the above finding makes clear that the assumption of time-varying cost of capital alone is not sufficient to provide a theoretical explanation for the observed excess volatility. Rather, it points to a fundamental discrepancy between the empirical results and the theoretical implications of the Gordon–Shapiro equation.

Our second consideration concerns the assumption itself. From today's perspective, it seems reasonable to abandon the problematic assumption that the cost of capital is strictly deterministic. Such an approach, however, requires the formulation of additional assumptions, the detailed discussion of which would go beyond the scope of this excursus. Methodologically, this implies that cost of capital $\widetilde{k}_t$ would have to be modeled as a stochastic process, fundamentally altering both the dynamics of the valuation formulas and the interpretation of empirical results.

What is crucial is the insight that the prerequisites required for this do not call into question the fundamental valuation principles explicitly formulated in the preceding chapter. It is therefore possible to work with stochastic cost of capital while still ensuring both the absence of arbitrage opportunities and transversality. This guarantees that the resulting prices are uniquely determined and that no "bubble solutions" arise.

The main result of the theory of stochastic cost of capital is a theorem, analogous to the Gordon–Shapiro theorem, which states that the present value of the firm is a stochastic rather than a deterministic multiple of the current cash flow:

$$\widetilde{V}_t^u = \frac{\widetilde{CF}_t^u}{\kappa_t} ,$$

where κ_t represents a different random variable. If it is not perfectly correlated with $\widetilde{CF}_t^u$, excess volatility already arises, since the relationship set out above no longer holds.

These approaches are indeed documented in the literature,[16] but they have so far met with only limited resonance and therefore cannot yet be regarded as established. Their systematic reception and further development remain tasks for future research. This excursus thus comes to a close.

2.1.5 Examples and Problems

We continue our analysis from Sect. 1.4.6.

[16] See Kruschwitz and Löffler (2020).

As it turns out, the cash flows in both of our examples, the finite one as well as the infinite one, satisfy the martingale-like condition. In the finite example, we showed this at the end of Sect. 1.3.3.1; in the infinite example, it appears in Eq. (1.7).

In the earlier Example sections, we primarily substituted numerical values into equations that had already been derived. Here, we take a more general approach and introduce a new relationship for the risk-neutral probabilities, which applies specifically within a binomial model. Although this relationship is not required for valuing the unlevered firm, it is nevertheless instructive to derive it explicitly. Our reasoning applies equally to both the finite and the infinite versions of the binomial model.

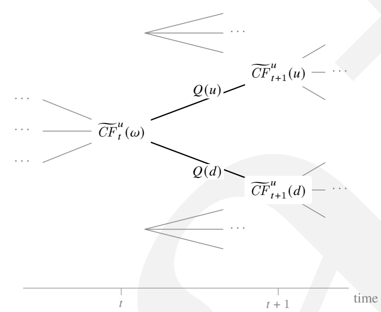


Fig. 2.2 Evaluation of Q at a local node with binomial branching.

Let ω be an arbitrary node at time t in our tree. We assume an arbitrary tree. What matters, however, is that starting from the node ω, the branching is locally binomial. After ω, there is only one up state and one down state, and no further immediate alternatives, regardless of what happens elsewhere in the tree; see Figure 2.2. In the following, we suppress the explicit reference to this node in order to simplify notation and improve readability. Accordingly, whenever we write cash flows such as $\widetilde{CF}_{t+1}(u)$, $\widetilde{CF}_{t+1}(d)$ or conditional probabilities $Q_{t+1}(u)$ and $Q_{t+1}(d)$, they are always to be understood as conditional on the fixed node ω; the same holds for the cash flow $\widetilde{CF}_t$.[17] In particular, $\widetilde{CF}_{t+1}(u)$ denotes the cash flow associated with an up-move from node ω, and $\widetilde{CF}_{t+1}(d)$ is interpreted analogously. The same convention applies to the corresponding probabilities. Hence, although ω is omitted from the notation, all subsequent expressions are to be read as conditional on this given node.

We seek the conditional probabilities $Q_{t+1}(u)$ and $Q_{t+1}(d)$ for these two possible movements. Naturally, the two conditional probabilities must sum to 100%, that is,

$$Q_{t+1}(u) + Q_{t+1}(d) = 1 \, .$$

However, another equation is needed in order to determine both quantities. Here, Theorem 2.3 proves useful, as it provides insight into how discount factors can be derived. This theorem, in turn, involves both conditional probabilities, thereby yielding

[17] In Eq. (1.6), we used the more explicit notation $Q_{t+1}(u|\omega)$ and $Q_{t+1}(d|\omega)$ for this conditional probability.

the second equation required for their determination. It states that at time t, the valuation of a single cash flow $\widetilde{CF}^u_{t+1}$ can be carried out in two equivalent ways, both leading to the same result:

$$\frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f} = \frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{1+k^{E,u}}\ .$$

From the martingale property of the cash flows (Assump. 2.1), we know that the conditional expectation can be expressed more simply. Taking into account that in the binomial model only two movements (up and down) are possible when evaluating the expectation on the left-hand side, the equation can be rewritten as:

$$\frac{Q_{t+1}(u)\,\widetilde{CF}^u_{t+1}(u) + Q_{t+1}(d)\,\widetilde{CF}^u_{t+1}(d)}{1+r_f} = \frac{(1+g)\widetilde{CF}^u_t}{1+k^{E,u}}\ .$$

We now have two linear equations in the conditional probabilities and can therefore solve the system explicitly. This yields:

$$Q_{t+1}(u) = \frac{\frac{1+r_f}{1+k^{E,u}}(1+g) - \frac{\widetilde{CF}^u_{t+1}(d)}{\widetilde{CF}^u_t}}{\frac{\widetilde{CF}^u_{t+1}(u)}{\widetilde{CF}^u_t} - \frac{\widetilde{CF}^u_{t+1}(d)}{\widetilde{CF}^u_t}},$$

$$Q_{t+1}(d) = \frac{\frac{\widetilde{CF}^u_{t+1}(u)}{\widetilde{CF}^u_t} - \frac{1+r_f}{1+k^{E,u}}(1+g)}{\frac{\widetilde{CF}^u_{t+1}(u)}{\widetilde{CF}^u_t} - \frac{\widetilde{CF}^u_{t+1}(d)}{\widetilde{CF}^u_t}}\ . \qquad (2.5)$$

At first glance, this representation may seem rather cumbersome. However, the two fractions it contains are straightforward to interpret. The term $\widetilde{CF}^u_{t+1}(u)/\widetilde{CF}^u_t$ represents the growth factor associated with an upward movement originating from the node $\widetilde{CF}^u_t$. Analogously, $\widetilde{CF}^u_{t+1}(d)/\widetilde{CF}^u_t$ refers to the corresponding factor in the case of a downward movement.

We will now apply this general formulation to our two examples.

2.1.5.1 The Finite Case (Continued)

Here, we suppose a risk-free interest rate of r_f =10% and consider a particular time, for example $t = 3$ and $\omega = dd$. As we have remarked already in Sect. 1.3.3.1 the cash flows in this example are martingale-like with a growth rate $g = 0$. We apply (2.5) and get

$$Q_3(u|dd) = \frac{\frac{1+0.1}{1+0.2}(1+0.1) - \frac{48.4}{88}}{\frac{145.2}{88} - \frac{48.4}{88}}, \quad Q_{t+1}(d|dd) = \frac{\frac{145.2}{88} - \frac{1+0.1}{1+0.2}(1+0.1)}{\frac{145.2}{88} - \frac{48.4}{88}}\ .$$

resulting in

$$Q_3(u|dd) \approx 0.4167, \qquad Q_3(d|dd) \approx 0.5833\ .$$

Using this idea at any time t and any available state ω we can finally determine all conditional probabilities. We have summarized our results in Fig. 2.3. It is evident that the particular probabilities depend on the node ω.

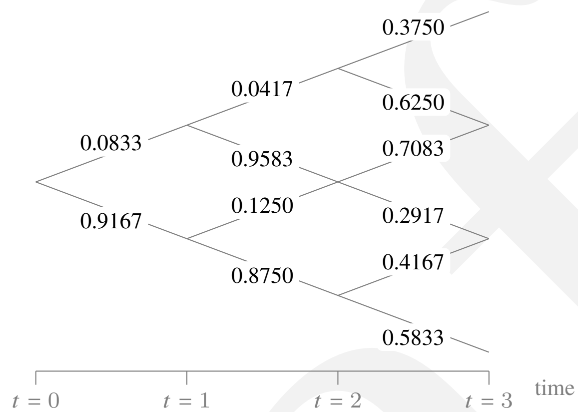


Fig. 2.3 Conditional Q-probabilities in the finite example.

2.1.5.2 The Infinite Case

As in the finite example we can evaluate the conditional up- and down-probabilities. To this end we assume that $r_f = 10\%$.

Applying (2.5), we observe that the growth rates of the cash flows are independent of both time and state. Throughout, we adopt the simplified notation from the previous subsection and omit the explicit reference to node ω,

$$u = \widetilde{CF}^u_{t+1}(u)/\widetilde{CF}^u_t, \quad d = \widetilde{CF}^u_{t+1}(d)/\widetilde{CF}^u_t \ .$$

Accordingly, for the infinite example we obtain

$$Q_{t+1}(u) = Q(u) = \frac{\frac{1+r_f}{1+k^{E,u}} - d}{u-d}, \qquad Q_{t+1}(d) = Q(d) = \frac{u - \frac{1+r_f}{1+k^{E,u}}}{u-d} \tag{2.6}$$

which holds regardless of t and ω.

This is an interesting result. First, the probabilities now do not depend on the state ω, therefore we will omit the node from now on. Furthermore, the factors u and d cannot be chosen arbitrarily if the cost of capital is constant. We can see that

$$d < \frac{1+r_f}{1+k^{E,u}} < u \tag{2.7}$$

must hold in order to ensure positive Q-probabilities. Any increase of the cost of capital might enforce a decrease of d, for example.

2.1.5.3 Problems

Problem 2.1 Consider cash flows that are independent of each other and identically distributed. As we have already mentioned in this case the conditional expectation

$$\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right], \quad \mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right], \ldots$$

will always be a real number.[18] Also, the cost of capital shall not be equal to the risk-free rate, $k_t \neq r_f$. We define discount rates k'_t as

$$\frac{\mathrm{E}^Q_t[\widetilde{CF}^u_{t+1}]}{1+r_f} =: \frac{\mathrm{E}_t[\widetilde{CF}^u_{t+1}]}{1+k'_t} .$$

Show that the value of the company is given by $V^u_t = \frac{\mathrm{E}^Q_0[\widetilde{CF}^u_1]}{r_f}$. Show that the cost of capital satisfies $k_t = r_f \frac{\mathrm{E}_0[\widetilde{CF}^u_1]}{\mathrm{E}^Q_0[\widetilde{CF}^u_1]}$. Verify that the the discount rate k'_t is not equal to the cost of capital.

Problem 2.2 We show that in a binomial model, the assumption of martingale-like cash flows admits a very simple interpretation.

Consider the cash flows from the finite example, Fig. 1.2. Show that the expected growth rate of the cash flows is always the same when measured at *each individual node*.

Problem 2.3 Assume that the cost of capital $k^{E,u}$ is deterministic and time-independent. The firm is perpetual ($T \to \infty$). Assume that the cash flows of the unlevered firm are martingale-like as in Assump. 2.1 for deterministic and constant g with $-1 < g < k^{E,u}$.

Show that the free cash flows are furthermore martingale-like under Q as well, i.e.,

$$\mathrm{E}^Q\left[\widetilde{CF}^u_{t+1}\right] = \left(1+g^Q\right)\widetilde{CF}^u_t .$$

and determine g^Q.

Problem 2.4 A straightforward extension of martingale-like cash flows would be to assume

$$\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] = \widetilde{CF}^u_t + C$$

for constant $C \neq 0$ (see for example Barberis et al. (1998), although they consider a different approach). Several problems will be devoted to this special case.

a) Prove that the infinitely lived unlevered firm having constant cost of capital satisfies

$$\widetilde{V}^u_t = \frac{\widetilde{CF}^u_t}{k^{E,u}} + \frac{1+k^{E,u}}{(k^{E,u})^2} C .$$

Hint: You might want to use

[18] See Sect. 2.1.2.

$$\sum_{s=1}^{\infty} \frac{s}{(1+x)^s} = \frac{1+x}{x^2} \qquad \text{if } x > 0\ .$$

b) Show that the expected capital gains rate of the unlevered firm is not zero.
c) Show that

$$\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u\right] = \frac{1+r_f}{1+k^{E,u}}\,\widetilde{CF}_t^u + \frac{r_f}{k^{E,u}}\,C$$

for the expectation of the cash flow under Q. Does Thm. 2.3 still hold?

Problem 2.5 Another straightforward extension of martingale-like cash flows would be that for every t

$$\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u\right] = \widetilde{CF}_t^u + X_t$$

where X_t is a random variable satisfying

$$\mathrm{E}_{t-1}\left[X_t\right] = \mathrm{E}_{t-1}^Q\left[X_t\right] = 0$$

and furthermore X_t is uncorrelated to $\widetilde{CF}_t^u$. Hence, this random variable is white noise and has no price at $t-1$. Several problems will be devoted to this special case.

a) Assume that the firm will live forever ($T \to \infty$). Verify that the value of the company having constant cost of capital satisfies

$$\widetilde{V}_t^u = \frac{\widetilde{CF}_t^u}{k^{E,u}} + \frac{X_t}{k^{E,u}}\ ,$$

and show that the variance of the firm value $\widetilde{V}_t^u$ is strictly greater than the variance of the corresponding cash flow $\widetilde{CF}_t^u$ if $k^{E,u} < 100\%$.
b) Verify that

$$\mathrm{E}_t\left[\widetilde{V}_{t+1}^u\right] = \widetilde{V}_t^u$$

and hence the expected capital gains rate is zero.
c) In this particular case the cost of capital may be used as discount rates (Thm. 2.3). Verify this by showing that

$$\frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u\right]}{1+k^{E,u}} = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u\right]}{1+r_f}$$

and

$$\frac{\mathrm{E}_t\left[\widetilde{CF}_{t+2}^u\right]}{(1+k^{E,u})^2} = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+2}^u\right]}{\left(1+r_f\right)^2}\ .$$

2.2 Basics about Levered Firms

We now bring to a close the discussion of unlevered firms and turn to the more realistic case of the levered firm. To do so we first of all need a clear separation between equity and debt. In addition, we will work out how the taxation of levered firms differs from that of the unlevered firm. These differences in taxation affect the value of the firm. And the degree of influence on value is dependent upon the type of financing policy the managers of the firm to be valued are operating under. Within this framework, we will analyze how numerous conceivable forms of financing policy effect value of firms and derive each appropriate valuation equation.

2.2.1 Two Types of Capital

We want to state some very basic considerations relevant to this section. A firm's capital can take various forms, such as common or preferred equity, corporate or convertible bonds, subordinated debt, and bank loans, to name just a few. In the DCF approach, however, we distinguish only two types of capital: equity and debt. In our discussion so far, the focus has been on equity.

Equity Equity is generally provided on a permanent basis, although investors may sell their shares at the current market value. It is inherently risky and grants its holders a claim to a stochastic cash flow. The DCF approach is primarily concerned with determining the value of a firm's equity.

It is not always possible to finance a firm entirely with equity. Moreover, corporate tax systems often create incentives to raise debt capital.[19] Debt differs fundamentally from equity.

Debt First, it is generally considered safe and is assumed to offer a risk-free rate of return. We have already noted that we assume this rate to be constant over time. Later, we will also consider what happens under insolvency.

Furthermore, we emphasize that the relevant assumption is not that debt must literally be issued for only one period. Debt may very well be issued under multi-period contracts. The point, rather, is that such claims are assumed to be tradable. Even when the original lender is contractually locked in, the claim itself can be sold in the market. In this way, debt is effectively one-period in nature, since at every point in time the market revalues and reallocates the claim.

Of course, this does not preclude debt financing from being continuously available in practice. What the present framework abstracts from are effects that arise specifically from fixed maturities beyond one period.

The firm's total net profit is stochastic and is allocated among its financiers such that debt financiers (creditors, debt holders) are served first, while equity financiers

[19] It is not uncommon to regard the valuation of the tax advantage of debt as a core task of the DCF approach.

(owners, shareholders) receive whatever remains. No additional categories of financiers are considered. The distribution rule itself is entirely straightforward.

Given debt path There are many possible debt paths, and in our framework they are taken as exogenous—given "from the sky," so to speak. We therefore do not select among them at this stage: before any optimization, we first need to compute what debt delivers (its advantages and disadvantages) and to do so we must hold the debt path fixed. We cannot and do not intend to decide here which of these debt paths is closest to reality or even in some sense optimal. Our aim is instead to lay out the range of conceivable ways in which debt might be taken as given, and to show how valuation works once such an exogenous debt path is imposed. Which path a firm ends up following is shaped by its credit arrangements with financiers and by managers' objectives and judgment.

In particular, we will not discuss which of the financing policies maximizes the value of the levered firm. Later it will become apparent that, with increased leverage, firm value rises. Hence, if they act rationally, the owners should choose leverage as high as possible. The owners will pursue a prespecified financing policy without worrying about whether this policy is optimal.

Assumption 2.2 (Given debt path) *Although future debt may be uncertain, the firm's debt policy is specified at $t = 0$ and known.*

Assump. 2.2 should be understood as an expression of the point-in-time principle. At the valuation date, the financing policy is not something we adapt to circumstances or adjust; it is part of the situation we face. We therefore take the debt path as given and ask what valuation follows once that path has been fixed.

We have already noted that the financiers can be divided into two groups. The financiers provide with capital, which the managers deploy in risky investments. In return, the financiers get securities which we term debt or equity, as the case may be. Although we presume our choice of words is already sufficiently clear, we do, however, want to get down an important characteristic of the securities. As was already suggested, we assume that equity and debt are traded on capital markets, that is, they can be bought and sold at any time. The securities thus have market prices, and we designate the market value of the equity at time t with $\widetilde{E}_t$, and the market value of the debt with $\widetilde{D}_t$. The tilde over the symbol makes it clear that a random variable is being dealt with. If we want to express that there are no random variables, we write D_t. Interest paid at time $t+1$ is $\widetilde{I}_{t+1}$.

Notation In the equations that we have used until now, we were always dealing with free cash flows and values of firms. In the previous chapter, when we considered unlevered firms, we added the index u to the required symbols. We now use the index l, if levered firms are being dealt with. We thus write $\widetilde{CF}^l_t$. For the equity cost of capital of the levered firms we will use the symbol $\widetilde{k}^{E,l}_t$.

We have to introduce a whole range of other symbols. It will always be recognized when these symbols are used in the context of an unlevered firm by index u. If, on the other hand, a levered firm is being dealt with, we will make that clear with the index l.

The Firm's Market Value, Debt–Equity Ratio, and Leverage Ratio The goal of our theory is the establishment of the market value of a firm. For the market value of the levered firm at time t, we use the symbol $\widetilde{V}_t^l$. The market value of the firm is equal to the sum of the equity's market value and the debt's market value,

$$\widetilde{V}_t^l := \widetilde{E}_t + \widetilde{D}_t \ .$$

Debt equity ratios and leverage ratios will play an important role in our further considerations. The debt ratio measures the proportion of debt to the market value of the firm,

$$\widetilde{l}_t := \frac{\widetilde{D}_t}{\widetilde{V}_t^l} \ , \tag{2.8}$$

while the leverage ratio (debt-equity ratio) is defined in the form

$$\widetilde{L}_t := \frac{\widetilde{D}_t}{\widetilde{E}_t} \ . \tag{2.9}$$

Even though we will use the debt ratios in later sections, we can also apply the leverage ratio. Since both quantities can easily be converted into each other this is not a limitation,

$$\widetilde{L}_t = \frac{\widetilde{l}_t}{1 - \widetilde{l}_t}$$

is valid. With these symbols we stress that all quantities are measured in market values.

Book Values In a few sections of this chapter, we consider the book values of equity and debt. These are the amounts at which owners' and creditors' claims are recorded on the firm's balance sheet. As symbols for debt and equity for book values, we use $\underline{\widetilde{D}}_t$ and $\underline{\widetilde{E}}_t$ respectively. The sums of equity and debt at time t are written in the form

$$\underline{\widetilde{V}}_t^l := \underline{\widetilde{E}}_t + \underline{\widetilde{D}}_t \ .$$

We notate

$$\underline{\widetilde{l}}_t := \frac{\underline{\widetilde{D}}_t}{\underline{\widetilde{V}}_t^l} \tag{2.10}$$

for the debt ratio measured in book values, and

$$\underline{\widetilde{L}}_t := \frac{\underline{\widetilde{D}}_t}{\underline{\widetilde{E}}_t}$$

for the correspondingly measured leverage ratio. Again, debt ratio and leverage ratio can easily be converted into each other.

2.2.2 Debt, Earnings and Taxes

In the foundational chapter of this book, we gave a preliminary introduction of gross cash flows and free cash flows. The terms developed there were perfectly adequate to come up with valuation equations for firms where the financing policy was not set in detail (Chapt. 1) or which were completely financed with equity (Sect. 2.1). Now that we are dealing with levered firms, we must introduce more structure into our terminology.

Tax Equation To clarify the relevant relationships, we again present the derivation of free cash flows shown in Fig. 1.1, now with the more detailed notation used in this chapter, see Fig. 2.4. Interest on debt is of course only accrued by levered firms. If we

Fig. 2.4 From earnings before taxes to free cash flow.

	Earnings before taxes	$\widetilde{EBT}$
+	Interest	$\widetilde{I}$
=	Earnings before interest and taxes	$\widetilde{EBIT}$
+	Accruals	$\widetilde{Accr}$
=	Gross cash flow before taxes	$\widetilde{GCF}$
−	Corporate income taxes	$\widetilde{Tax}$
−	Investment expenses	$\widetilde{Inv}$
=	Free cash flow	$\widetilde{CF}^{l}$

add it to the earnings before taxes (EBT) then we get the earnings before interest and taxes (EBIT). If we add accruals, we arrive at the gross cash flow before taxes, also called EBITDA. If we finally deduct the investment expenses and deduct taxes as well, then we get the free cash flow. This cash flow is fully distributed to the owners of the company.

As announced earlier, we restrict attention in this chapter to corporate taxation and omit income taxation at the shareholder level. The tax base for profit taxes is earnings before taxes (EBT)

$$\widetilde{Tax} = \tau\,\widetilde{EBT}\;. \tag{2.11}$$

The tax equations should be valid independent of the sign of the tax base. If the tax base is positive, the firm has to pay taxes; if it is, on the other hand, negative, then the firm gets a return in the amount of the tax due. We will not give more realistic models of loss set-off or loss carryback rules than that. We do so because we assume a proportional corporate income tax; under proportional taxation, profits and losses net one-for-one, so this stylized treatment suffices for our purposes.

In order to describe the difference between the levered and the unlevered firm we will assume that their investment policies coincide.

Assumption 2.3 (Identical gross cash flows) *The gross cash flows before taxes as well as the accruals and investment expenses of the unlevered firm do not differ from those of the levered firm.*

Hence, the EBIT must be just as large for the firm financed by equity as it is for the firm financed by debt. In Fig. 2.4 the third and the fourth line are thus identical for the levered and unlevered firms, while taxes and free cash flows will show different values. The taxes of the levered firm are thus smaller by the product of the tax rate and interest on debt than the taxes of the firm financed solely by equity

$$\widetilde{Tax}_t^u - \widetilde{Tax}_t^l = \tau \, \widetilde{I}_t \; .$$

Financing by debt is thus favored in this model.

By Assump. 2.3 the free cash flows before taxes of the levered and unlevered firms are identical, only the tax payments are different,

$$\widetilde{CF}_t^l = \widetilde{CF}_t^u + \tau \, \widetilde{I}_t \; . \tag{2.12}$$

The firm financed by equity has lower free cash flows than the firm financed by debt, because interest may be deducted from the tax base. The following concerns the question as to the value of these tax advantages. A (if not *the*) central problem of the DCF theory is the determination of the value of the tax advantages governed by credit conditions.

2.2.3 Financing Policies

No Insolvency We are supposing at first that credit is not threatened by insolvency and thus follow a widely held tradition within DCF literature,

$$\widetilde{I}_t = r_f \, \widetilde{D}_{t-1} \; . \tag{2.13}$$

This notion flagrantly contradicts the experience of banks with their borrowers. In reality financiers' claims are obviously under notable threat of insolvency. In a later subsection we will show how insolvency can be handled in our theory.

Components of Tax Advantages Now we make a first attempt at turning to the valuation of tax advantages of the firm financed by debt. The tax advantages we just determined arise because interest on debt may be deducted from the firm's tax base. This comes to a tax savings—also called tax shield—in the amount of

$$\tau \, r_f \, \widetilde{D}_{t-1} \; .$$

Anyone professionally involved in the valuation of firms—be it as a certified public accountant, an investment banker, or a business consultant—knows only too well that, in practice, both tax rates and interest rates are uncertain. In our assumptions, however, uncertainty is confined to the amount of debt $\widetilde{D}_{t-1}$.

The interest rate of debt r_f as well as the tax rate τ are risk-free according to our requirements.

Value of Tax Advantages To value tax advantages appropriately, we need further information about the uncertainty to which they are exposed. It is true that, under our assumptions, we know the tax rate and the interest rate. However, without further assumptions about the financing policy at time $t = 0$, we cannot determine the firm's debt level at time $t > 0$.

We have no other options than to come to further assumptions regarding information about the firm's future debt policy. This is the only way we can determine the amounts of debt $\widetilde{D}_{t-1}$ outstanding at dates $t > 0$, the risk associated with those amounts, and, accordingly, how the resulting tax advantages should be valued. Without any information of the levered firm the tax advantages cannot be properly identified.

The practitioners may be perplexed to note that firm valuation depends on its underlying assumptions. In so doing the firm value takes on an air of doing what one pleases of it, or—to put it more drastically—an element of manipulation. To that we must answer, every valuation is based on expectation about the future. Whoever does not forecast the turnover numbers, whoever does not know what the cost of materials will be, cannot value a firm. All of these and further assumptions come off as somewhat arbitrary. That also applies for the financing policy of the analyst.

Different Financing Policies Now let us turn to different possible financing policies. We find that within this area of DCF literature, two concepts are regularly brought into play. Autonomous financing supposes that the amounts of debt are already fixed at the time of valuation. Financing based on value supposes, in contrast, that the debt ratios are fixed in the present.

We too will examine both financing policies. In what follows, when debt ratios are measured at market values, we will, for precision, refer to this as a financing policy based on market values. Moreover, we will bring four further financing policies into the discussion. We term these policy based on cash flows, policy based on dividends, policy based on book values and on cash flow-debt ratio. These six policies can thus be characterized in brief by the following:

1. With *autonomous financing methods*, the future amount of debt is deterministic.
2. With the *financing based on market values*, the analyst sets the future debt ratios based on market values.
3. With the *financing based on book values*, the future debt ratios are not fixed to market values, but rather to book values.
4. With the *financing based on cash flows* the amount of debt is based on the firm's cash flows.
5. With the *financing based on dividends* the firm's amount of debt is managed so that a previously determined dividend can be distributed.

6. With the *financing based on dynamical leverage ratio* the analyst sets the future cash flow-debt ratios.

To determine firm values under different financing policies, we need an important equation. The statements of the Fundamental Theorem apply to both levered and unlevered firms,[20] in particular the valuation results given in Thm. 1.2. Thus, the value of the unlevered firm is given by the equation

$$\widetilde{V}_t^u = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u\right]}{1+r_f} + \ldots + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_T^u\right]}{\left(1+r_f\right)^{T-t}} .$$

The validity of the Fundamental Theorem does not depend upon how the firm is financed. The value of the levered firm is thus given by the relation

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^l\right]}{1+r_f} + \ldots + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_T^l\right]}{\left(1+r_f\right)^{T-t}} .$$

Using (2.12) it follows immediately from both valuation equations that the market value of the levered firm differs from that of the unlevered firm only by the value of the tax advantages,

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{\mathrm{E}_t^Q\left[\tau \widetilde{I}_{t+1}\right]}{1+r_f} + \ldots + \frac{\mathrm{E}_t^Q\left[\tau \widetilde{I}_T\right]}{\left(1+r_f\right)^{T-t}} .$$

With (2.13) and Rule 2 (Linearity) this yields

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{\tau r_f\, \mathrm{E}_t^Q\left[\widetilde{D}_t\right]}{1+r_f} + \ldots + \frac{\tau r_f\, \mathrm{E}_t^Q\left[\widetilde{D}_{T-1}\right]}{\left(1+r_f\right)^{T-t}} . \tag{2.14}$$

We will be able to use this equation for all financing policies. Fernández (2005) gives the same result in the following presentation,[21]

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \tau \widetilde{D}_t + \frac{\tau\, \mathrm{E}_t^Q\left[\widetilde{D}_{t+1} - \widetilde{D}_t\right]}{1+r_f} + \ldots + \frac{\tau\, \mathrm{E}_t^Q\left[\widetilde{D}_T - \widetilde{D}_{T-1}\right]}{(1+r_f)^{T-t}} . \tag{2.15}$$

2.2.4 Debt and Transversality (Again)

In the following we will focus on levered companies that accept loans which run forever. Anyone familiar with basic financial mathematics knows that the present value of a risk-free loan equals the sum of discounted cash flows if the credit is temporary. So if credit is granted today in the amount of $\widetilde{D}_0$ and the borrower in later times pays both

[20] See Sect. 1.4.4.

[21] See Prob. 2.8.

interest of $r_f \widetilde{D}_t$ and (possibly negative) redemptions of $\widetilde{D}_t - \widetilde{D}_{t+1}$, then

$$\widetilde{D}_0 = \sum_{t=0}^{T} \frac{(1+r_f)\widetilde{D}_t - \widetilde{D}_{t+1}}{(1+r_f)^t}$$

must hold if $\widetilde{D}_T = 0$ is assumed.

What will happen if $T \to \infty$? One can easily imagine that the loan is not repaid in full. But what does that mean for the above equation? Under what conditions can we state

$$\widetilde{D}_0 = \lim_{T\to\infty} \sum_{t=0}^{T} \frac{(1+r_f)\widetilde{D}_t - \widetilde{D}_{t+1}}{(1+r_f)^t} , \tag{2.16}$$

and under what conditions is this not permissible? Transversality, again, will give the answer. For this reason, we reassume this condition. However, we will concentrate on uncertainty from the beginning. It is immediately apparent that this is an assumption about an infinite-horizon financing policy.

Assumption 2.4 (Debt policy satisfies transversailty) *The debt policy of the firm satisfies the transversality condition, if for all t*

$$\lim_{T\to\infty} \frac{\mathrm{E}_t^Q[\widetilde{D}_T]}{(1+r_f)^{T-t}} = 0 .$$

To fully understand this assumption, let us consider two examples of infinite financing policies (extreme, in our view). In the first case, transversality will be violated; in the second, it will be satisfied. Readers must decide whether the examples are intuitively clear.

First, assume that the payment obligations due to interest liabilities will be completely financed by credit. This boils down to the relation

$$\widetilde{D}_{t+1} = (1+r_f)\widetilde{D}_t ,$$

and it is immediately clear that the transversality condition can not be fulfilled: The creditor increases his loan from year to year and never gets a refund. Eq. (2.16) can therefore never be satisfied for a positive $\widetilde{D}_0$. In finite time the credit might grow enormously, but would eventually be repaid.

The next example shows that Eq. (2.16) can be sustained with a financing policy that is only slightly different from the previous one. To this end, let us assume that an investor finances only half of the interest obligations by a credit. Again, the loan is increasing year by year. However, the new policy is now described by

$$\widetilde{D}_{t+1} = \left(1 + \frac{r_f}{2}\right) \widetilde{D}_t .$$

Anyone who believes that Eq. (2.16) is not fulfilled under this policy is wrong. This follows directly from

$$\lim_{T\to\infty} \frac{\mathrm{E}_t^Q[\widetilde{D}_T]}{(1+r_f)^{T-t}} = \lim_{T\to\infty} \frac{\left(1+\frac{r_f}{2}\right)^T}{(1+r_f)^{T-t}} \widetilde{D}_0$$
$$= \lim_{T\to\infty} (1+r_f)^t \widetilde{D}_0 \underbrace{\left(\frac{1+\frac{r_f}{2}}{1+r_f}\right)^T}_{<1} = 0 \,.$$

The difference between the two financing policies is readily apparent. For Eq. (2.16) to be satisfied, the lender must receive some payments from the debtor at every point in time. The compounding effect then ensures complete refunding of the loan. However, if the redemption payments go back to zero (as in the first example), the transversality condition is inevitably violated.

The issue raised here matters only when a revolving credit line—although written as a sequence of one-period loans—is renewed indefinitely. Ultimately, the assumption implies that in an infinite-horizon setting, part of the debt or at least part of the interest must be serviced regularly; otherwise, meaningful results cannot be obtained. Therefore, we will treat $T \to \infty$ only exceptionally, and will restrict ourselves to contracts with finite duration.

2.2.5 Problems

Problem 2.6 Show that the recursion (2.20) implies

$$\widetilde{D}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{I}_s + \widetilde{Pr}_s\right]}{(1+r_f)^{s-t}} \,.$$

Problem 2.7 A levered firm lives until $T = 3$. Assume that the conditional Q-probability of the up movement is 0.25 and the risk-free rate is 10% , the tax rate is 50%. Evaluate the tax shield $\widetilde{V}_0^l - \widetilde{V}_0^u$ if the debt schedule is as in Fig. 2.5.

Problem 2.8 Prove that from (2.14) it follows that

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \tau \widetilde{D}_t + \sum_{s=t+1}^{T} \frac{\tau\, \mathrm{E}_t^Q\left[\widetilde{D}_s - \widetilde{D}_{s-1}\right]}{(1+r_f)^{s-t}} \,.$$

This is the main result in Fernández (2005).

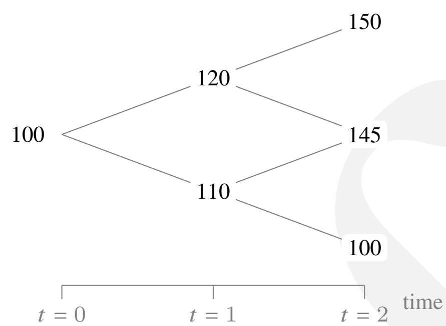


Fig. 2.5 Debt in Prob. 2.7.

2.3 Insolvency

2.3.1 Insolvency and Valuation

Authors working on credit risk typically discuss insolvency probabilities in detail. Surprisingly, the same considerations are often disregarded in the DCF literature. There is no doubt that we must address how to value a firm when there is a risk that it will be unable to meet all of its debt obligations. It will turn out that, for a levered firm, insolvency becomes much more multifaceted than in the unlevered case.

We have already noted that we will focus exclusively on a structural model of insolvency. Accordingly, to determine the conditions that lead to insolvency, we will look only at cash flows, firm value, and their stochastic properties. We will not introduce an additional stochastic process—separate from cash flows and firm value—solely to model insolvency.

We now impose assumptions within our framework that are intended to remain valid even if insolvency occurs. In what follows, we briefly explain why this is a consistent and economically natural requirement.

Homogeneous Expectations Up to now we have worked on the basis that debt and equity financiers are equally well informed. This condition is of particular importance with the threat of insolvency and can surely be seen critically. However, there is no getting around providing debt financiers with some information about the firm. No one lends money without first carefully assessing the borrower's business ideas, risks, and market prospects.

Yet asymmetric information as a rule is truer to reality than our condition of homogenous expectations. It, however, applies that whoever wants to deviate from this assumption, has to very precisely define the information which both sides either have or do not have access to.

Identical Cash Flows and Insolvency We have consistently assumed that the gross cash flows of levered and unlevered firms are identical. So far, however, we have considered only levered firms that do not become insolvent. If we want to maintain this assumption, we must extend it to cases in which the firm is exposed to insolvency risk. In other words, we assume that the firm's gross cash flows remain the same regardless of whether or not insolvency risk is present.

This is certainly a far-reaching restriction, but we still regard it as a sensible one. Of course, a firm in financial distress may enter a situation in which its gross cash flows are affected. Financial difficulties often lead both suppliers and customers to reconsider whether they want to continue doing business with the firm. Some customers leave altogether or terminate long-term contracts, while suppliers may be willing to deliver only against prepayment. Managers who might have remained loyal to the firm under more favorable circumstances begin to look for other jobs, taking with them precisely the know-how that is most valuable in times of crisis and thereby worsening the situation further.

Even so, we find it useful to abstract from these effects. It is this business model that generates the cash flows, and for the purposes of our analysis we take it as given. We do so because we do not see how any generally valid statement could be made about the precise way in which an insolvency event would affect the firm's operations and, through them, its cash flows. Such effects are inevitably case-specific. Important as they may be in practice, they do not lend themselves to general theoretical analysis. What may change, by contrast, is the question of whether equity holders will ultimately receive those cash flows. But the cash flows themselves remain the unchanged basis of our calculations.

All the financial consequences of high leverage discussed above are usually referred to in the literature as indirect costs of default. In this sense, our model abstracts from such indirect default costs. These indirect costs are notoriously difficult to quantify. A more realistic treatment would therefore require, despite these difficulties, an explicit functional relationship between gross cash flows and leverage. In a multi-period model, gross cash flows have two dimensions: magnitude and timing. Our assumption is that insolvency affects neither.

The same applies, for the time being, to financing policy. In the cases considered first, we continue to work with Assump. 2.2 and thus keep financing policy unchanged, provided the firm still remains in the hands of its original equity holders. This is not because we regard such an assumption as universally realistic, but because in these cases there are good reasons to preserve the policy initially specified. Later, however, we will also consider a case in which insolvency leads to a change in ownership. Once the firm no longer belongs to its original owners, it is no longer plausible to assume that the previous financing policy will simply remain in place.

Given these assumptions, one may well ask what exactly we mean by "incorporating insolvency risk". We have ruled out changes in cash flows, and we have likewise set aside any adjustment in financing policy. So what, then, remains of insolvency risk in the model? The answer is straightforward: our analysis focuses exclusively on the effect that an insolvency trigger may have on the allocation of firm value between equity holders and creditors.

How much do the owners receive at time t? We begin with the firm's gross cash flow before interest and taxes, $\widetilde{GCF}_t$. To obtain free cash flow, we deduct the firm's internally financed investments and its taxes. The remaining question is whether these quantities differ between an unlevered firm and a levered firm that faces insolvency risk. As before, we address this by assuming that the investments and accruals of a firm that becomes insolvent are the same as those of a firm that does not:

Assumption 2.5 (Gross cash flows and insolvency) *The gross cash flows as well as the investment and accruals of the unlevered firm do not differ from those of the firm becoming insolvent.*

Bankruptcy Estate If the company does not file for bankruptcy, then the creditors' claims can be satisfied in full. Two parties are to be differentiated here, the state and the investor. The order in which the claims of the finance administration and the other creditors are satisfied does not matter as long as we are not dealing with insolvency. The owners' claims will be settled last in any case. In the worst case the shareholders can end up with nothing. Since corporations do not have personal liability, we can disregard the owners having to make payments from their private pockets in very unfavorable situations.

As a rule, the liquidation proceeds in bankruptcy do not suffice to fully satisfy the state and other creditors. Thus, priority matters: Is the state paid in full first, or are other creditors paid while the state waits? The answers are set by the relevant legal provisions. We do not pursue this further here; instead, we resolve the issue by adopting an assumption that is (more or less) satisfied in most industrial countries.

Assumption 2.6 (Prioritization of debt) *The tax authority's claims have priority over those of other creditors.*

The tax authority is therefore given priority over other creditors in the event of insolvency, so that tax claims can be satisfied in full. In our framework, insolvency is never so severe that the state loses any part of its claims.

Notation The notation used so far is insufficient for the analysis that follows. Let us again suppose that the firm took out the loan $\widetilde{D}_t$ at time t. In the earlier section the variable $\widetilde{D}_t$ signified two different things, namely, for one, the credit which the firm took in at time t, and, for the other, the amount, which apart from the interest it redeems at time $t+1$.[22] In case of insolvency, the amount, which the company amortizes at the time $t+1$, will not coincide with the repayment sum to which it is legally obligated. $\widetilde{D}_t$ shall be the credit which has been raised at time t and $\widetilde{D}_{t+1}$ the corresponding amount a year later. Consequently, the difference between $\widetilde{D}_t$ and $\widetilde{D}_{t+1}$ accounts for the amount which the company needs to pay back to the creditor (or, if this amount is negative, has to be raised). In the following we will assume that the company pays back the amount $\widetilde{Pr}_{t+1}$ (the principal) which can be at the most as high as $\widetilde{D}_t - \widetilde{D}_{t+1}$, hence

[22] See Sect. 2.2.1.

$$\widetilde{Pr}_{t+1} \leq \widetilde{D}_t - \widetilde{D}_{t+1} \ .$$

If the repayment sum is smaller than the amount which the company owns its creditors the term

$$\widetilde{D}_t - \widetilde{D}_{t+1} - \widetilde{Pr}_{t+1}$$

describes a remission of debts. We do not need a new symbol for the interest $\widetilde{I}_{t+1}$ resulting at time $t+1$.

Focusing solely on the relationship between the firm being valued and its financiers in insolvency, it does not matter how the remaining funds are allocated between interest and principal due. For tax purposes, however, the distinction matters: interest reduces the tax base, whereas principal repayment does not.

We assume that the tax authority allows interest $\widetilde{I}_{t+1}$ to be deducted from the tax base. Furthermore, in many countries the state requires that, in bankruptcy, cancellation of debt be taxed in the amount $\widetilde{D}_t - \widetilde{D}_{t+1} - \widetilde{Pr}_{t+1}$. According to (2.11), the following holds for the taxes of a levered firm that becomes insolvent at time t:

$$\widetilde{Tax}^l_{t+1} = \tau \left(\widetilde{GCF}_{t+1} - \widetilde{Accr}_{t+1} - \widetilde{I}_{t+1} + \widetilde{D}_t - \widetilde{D}_{t+1} - \widetilde{Pr}_{t+1} \right) \ .$$

Since the unlevered firm's tax equation does not change, and the gross cash flows and investments are identical, we now obtain

$$\begin{aligned} \widetilde{CF}^l_{t+1} &= \widetilde{GCF}_{t+1} - \widetilde{Inv}_{t+1} - \widetilde{Tax}^l_{t+1} \\ &= \widetilde{GCF}_{t+1} - \widetilde{Inv}_{t+1} - \widetilde{Tax}^u_{t+1} + \tau \left(\widetilde{I}_{t+1} - \widetilde{D}_t + \widetilde{D}_{t+1} + \widetilde{Pr}_{t+1} \right) \\ &= \widetilde{CF}^u_{t+1} + \tau \left(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t \right) \ . \end{aligned} \tag{2.17}$$

The Fundamental Theorem of Asset Pricing also applies to a levered firm that becomes insolvent. We can thus establish the following relation:

$$\begin{aligned} \widetilde{V}^l_t &= \sum_{s=t+1}^{T} \frac{\mathrm{E}^Q_t \left[\widetilde{CF}^l_s \right]}{(1+r_f)^{s-t}} \\ &= \widetilde{V}^u_t + \sum_{s=t+1}^{T} \frac{\tau \, \mathrm{E}^Q_t \left[\widetilde{I}_s + \widetilde{Pr}_s + \widetilde{D}_s - \widetilde{D}_{s-1} \right]}{(1+r_f)^{s-t}} \ . \end{aligned} \tag{2.18}$$

We now state a key proposition regarding the valuation of an insolvent firm.

Valuation of Insolvent Firms We have already noted that our model distinguishes between insolvency and default. Insolvency refers to a specific state of the firm without yet specifying its implications for equity and debtholders. Those implications are characterized by default.

This approach enables us to separate the triggers of insolvency from the redistribution mechanisms that arise upon default. When analyzing default, we need not know yet which particular trigger caused the insolvency; in this sense, the consequences can be determined independently of the cause.

For the purpose of redistribution, it is sufficient to note that creditors act rationally: they will extend credit only if total repayments are, in valuation terms, equivalent to the amount of the loan granted. Accordingly, the Fundamental Theorem of Asset Pricing also applies in this context. For any date s, if $\widetilde{Pr}_{s+1}$ is repaid at $s+1$, we have

$$\widetilde{D}_s = \frac{\mathrm{E}_s^Q\left[\widetilde{I}_{s+1} + \widetilde{Pr}_{s+1} + \widetilde{D}_{s+1}\right]}{1+r_f} \ . \tag{2.19}$$

Using Rule 5 (Known Factor) this gives us

$$r_f \widetilde{D}_s = \mathrm{E}_s^Q\left[\widetilde{I}_{s+1} + \widetilde{Pr}_{s+1} + \widetilde{D}_{s+1} - \widetilde{D}_s\right]$$

and with Rule 4 (Iterated Expectations) for all $s \geq t$ finally

$$r_f\, \mathrm{E}_t^Q\left[\widetilde{D}_s\right] = \mathrm{E}_t^Q\left[\widetilde{I}_{s+1} + \widetilde{Pr}_{s+1} + \widetilde{D}_{s+1} - \widetilde{D}_s\right] \ . \tag{2.20}$$

Entering in Eq. (2.18) results in

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \sum_{s=t+1}^{T} \frac{\tau r_f\, \mathrm{E}_t^Q\left[\widetilde{D}_{s-1}\right]}{\left(1+r_f\right)^{s-t}}$$

and this equation is no different from Eq. (2.14), in which we excluded bankruptcy risk!

As long as financing policy remains unchanged even in the presence of insolvency, the same valuation relations continue to apply as in the case without insolvency. Insolvency risk as such does not alter the valuation results. Only if it also leads to a change in financing policy, and thus in the associated tax shields, do valuation effects arise.

This becomes more intuitive once one recalls two features of our framework. First, we work with a structural model of insolvency: the states in which insolvency may occur are exactly those already contained in the firm's cash-flow dynamics; no additional stochastic driver is introduced for default. Second, cash flows remain unchanged, so insolvency affects only the distribution of firm value between equity holders and creditors. Redistribution alone does not change firm value. Put differently, insolvency in our model arises from debt, not from an additional source of real-side risk and as long as cash flows and financing policy remain unchanged, no value is created or destroyed.

The decisive assumption here is Assump. 2.2. A firm may well react to looming insolvency by changing its financing policy, and in practice this often happens. But such ex post revisions are not what we mean by a coherent incorporation of insolvency risk. Rather, financing policy must be specified from the outset in a way that already accommodates the possibility of insolvency. This does not mean that the composition of claims can never change; transfers of assets or debt-to-equity conversions may well occur. But if they do, they must be understood as consequences built into the original financing arrangement, not as ad hoc corrections introduced later.

This is best viewed as a concrete application of the point-in-time principle. The principle itself is abstract, but here it takes the form of a precise modeling requirement: the relevant financing arrangement must already contain a coherent account of what happens if insolvency risk materializes. The message of this subsection is therefore straightforward. The valuation of firms in the presence of insolvency risk does not fail because of any weakness in DCF theory. The real difficulty lies in formulating financing policies carefully enough, and in assessing whether they are feasible in the market.

2.3.2 Insolvency triggers

Coupon Rate If the creditors do not have to worry about insolvency, they will demand the risk-free rate. The situation might be different, however, when there is a risk of going bankrupt. In this case, the creditors face the possibility that their interest and principal payments may not be fully honored in every conceivable state of the world. To receive adequate compensation for this risk, they require borrowers to agree to a nominal interest rate higher than the risk-free rate. In what follows, this coupon rate will be denoted by c_t. It will be paid only if no insolvency occurs. What happens in the event of insolvency—beyond the general description of repayment as $\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1}$—must remain open at the moment. We merely note that debt holders likewise demand a fair deal; accordingly, the Fundamental Theorem of Asset Pricing applies to their payments as well. Figure 2.6 illustrates our approach.

Since the model does not specify the provisions governing the event of insolvency, we are at the moment not able to determine c_t completely. But we should expect $c_t > r_f$ when there is a positive probability of insolvency.

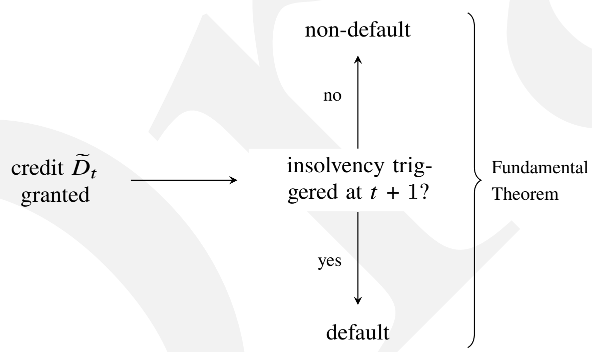


Fig. 2.6 Insolvency, default, and rational debt holders.

Cost of Debt It might be that in some states of the world the payments for interest and redemption lie below the risk-free rate and therefore the firm demands a higher interest rate in the remaining states. Analogous to the cost of equity this requires a definition of cost of debt. Someone who invests $\widetilde{D}_t$ today is entitled to payments amounting to $\widetilde{D}_t + \widetilde{I}_{t+1}$ less remission of debts. Due to a remission of debts of $\widetilde{D}_t - \widetilde{D}_{t+1} - \widetilde{Pr}_{t+1}$, we obtain the following definition.

Definition 2.3 (Cost of debt) *The cost of debt $\widetilde{k}_t^D$ of a levered firm is the conditional expected return*

$$\widetilde{k}_t^D := \frac{\mathrm{E}_t\left[\widetilde{D}_{t+1} + \widetilde{I}_{t+1} + \widetilde{Pr}_{t+1}\right]}{\widetilde{D}_t} - 1 \ .$$

However, unless there is the probability of insolvency there is no reason whatsoever to assume that the cost of debt is different from the risk-free rate or the coupon rate

$$\widetilde{k}_t^D = c_t = r_f \ .$$

Notice that we do not require the cost of debt to be deterministic today. Although this will be a necessary requirement for different types of cost of capital later,[23] cost of debt will not be used itself to determine the value of firms and hence need not to be deterministic.

Insolvency Triggers For Levered Firms We already introduced two insolvency triggers: illiquidity and over-indebtedness. While the first trigger relates to the firm's payment level, the second is based on the asset level. Illiquidity, therefore, simply means being unable to meet one's obligations as they fall due. Under our assumptions, such a trigger depends both on the amount of debt raised today and on receivables carried over from the previous period. In our model, for example, illiquidity could in principle be avoided by raising more debt today; however, doing so would violate our assumption that the debt path is predetermined and therefore not subject to change. The debt path is taken as given and is not adjusted in response to efficiency, optimality, or value-maximization considerations.

Until now, we have defined insolvency only for unlevered firms; we now extend this to levered firms. Whereas illiquidity focuses on cash flows, we use the term over-indebtedness when the equity value of the firm turns negative. In what follows, however, we assess over-indebtedness using market values rather than, for example, book values. The reason is straightforward: by using market values, we can state clear conceptual relationships and provide necessary and sufficient conditions for bankruptcy. That would be much more difficult—if not impossible—were we to use book values. Hence, relying on market values is consistent with the simplification strategies that prevail in economics.[24]

[23] See the subsections on TCF, FTE etc.

[24] German insolvency law (Insolvenzordnung) incorporates a further aspect of over-indebtedness that we abstract from here. DCF analysis focuses on the firm's equity value. Under Assumption 2.2, equity is determined with reference to current debt. German insolvency law, by contrast, is concerned with the creditors' "outstanding claims", which, in terms of the present model, may be understood as $(1+c)\widetilde{D}_{t-1}$ from the previous period $t-1$. Over-indebtedness in that sense arises only when these claims exceed the firm's current asset value. We do not adopt that definition here.

Definition 2.4 (Insolvency triggers) *For a given financing policy, a levered firm will be* illiquid at time $t+1$ in state ω *if the cash flows in this state do not suffice to fulfill the creditors' payment claims (interest and net redemption) as contracted,*

$$\widetilde{CF}^{l}_{t+1}(\omega) - \big((1+c_t)\widetilde{D}_t(\omega) - \widetilde{D}_{t+1}(\omega)\big) < 0 \,, \tag{2.21}$$

where c_t is the coupon rate owners agreed upon.

For a given financing policy, a levered firm will be over-indebted at time $t+1$ in state ω *if the market value of debt exceeds the firm's market value,*

$$\widetilde{V}^{l}_{t+1}(\omega) < \widetilde{D}_{t+1}(\omega) \,. \tag{2.22}$$

Notice that both definitions refer to a future date t and the state ω from today's perspective. It can easily be seen, that our definition generalizes the case of the unlevered firm.

We want to examine whether a firm that is over-indebted will become illiquid at a future point in time, and vice versa. We believe that the relationship between both triggers requires more attention than they are currently given in the literature.[25] Authors who address valuation problems seem to assume that it does not matter which insolvency trigger is used—a view we challenge although in Thm. 2.4 we have exactly shown that. For investors and financiers (e.g., in the context of insolvency–risk forecasting) it is essential to determine whether these triggers are interchangeable or whether one constitutes a stricter criterion in the sense that it is met earlier.

We prove that over-indebtedness always implies illiquidity, whereas the converse implication is more subtle.

Over-Indebtedness Implies Illiquidity Disregarding specific assumptions concerning the dynamics of the free cash flows, we can prove that over-indebtedness implies illiquidity. This result is immediately apparent. Just realize that debts represent the present value of cash outflows while assets represent the present value of cash inflows. Having said this, it must be that at some future point of time an outflow is greater than an inflow if debts exceed assets today.

For unlevered firms, the result is immediate. If the firm's market value is negative, then future cash flows must also be negative in at least some states. The same idea carries over to levered firms, as the following theorem shows.

[25] The following considerations are—in parts literally—taken from Kruschwitz et al. (2015) and Löffler and Steins (2025).

Theorem 2.5 (Over-indebtedness implies illiquidity) *Assume $c_t \geq r_f$. If a levered company is over-indebted at time t in some state, then there is a date $s \geq t$ and a state where the firm is illiquid.*

Notice that illiquidity does not necessarily imply over-indebtedness.[26]

We prove the theorem by contradiction. To this end, we consider an over-indebted firm that will never be illiquid. In this case the inequality

$$\widetilde{CF}^l_s(\omega) - \left((1+c_{s-1})\widetilde{D}_{s-1}(\omega) - \widetilde{D}_s(\omega)\right) \geq 0$$

applies for all states ω and times $s \geq t$. Multiplying the preceding inequality by the risk-neutral probabilities, using $c \geq r_f$ and summing up leads to

$$\mathrm{E}^Q_t\left[\widetilde{CF}^l_s\right] \geq \mathrm{E}^Q_t\left[(1+c_{s-1})\widetilde{D}_{s-1} - \widetilde{D}_s\right] \geq \mathrm{E}^Q_t\left[(1+r_f)\widetilde{D}_{s-1} - \widetilde{D}_s\right] \ .$$

Dividing by $(1+r_f)^{s-t}$ and adding up over all t results in

$$\begin{aligned} \sum_{s=t+1}^{T} \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^l_s\right]}{(1+r_f)^{s-t}} &\geq \sum_{s=t+1}^{T} \frac{\mathrm{E}^Q_t\left[\left((1+r_f)\widetilde{D}_{s-1} - \widetilde{D}_s\right)\right]}{(1+r_f)^{s-t}} \qquad \text{and using (2.20)} \\ &= \sum_{s=t+1}^{T} \frac{\mathrm{E}^Q_t\left[\widetilde{I}_s + \widetilde{Pr}_s\right]}{(1+r_f)^{s-t}} \end{aligned}$$

Now, (2.19) implies that this sum is nothing more than $\widetilde{D}_t$ because if debt $\widetilde{D}_t$ is granted the debtor will get $\widetilde{I}_s + \widetilde{Pr}_s$ for all $s > t$ (see also Prob. 2.6). Since the term on the left-hand side of the inequality is the firm value $\widetilde{V}^l_t$, this contradicts over-indebtedness. This was to be shown. □

Equity Transfer and Insolvency Triggers On the basis of the statement above, one might be tempted to conclude that this exhausts the connection between the two insolvency triggers; in particular, one might suspect that the converse dependence (namely, that illiquidity entails over-indebtedness) does not hold in general. While such a suspicion is formally correct, the matter is more subtle.

For the following considerations, we go one step further than before (see, in particular, Sect. 2.3.2). Up to that point, we addressed default only at a very general level and did not go into its contractual details. There is a compelling reason for this: once insolvency occurs, there is not just one, but a whole range of possible ways in which a firm's creditors may respond to the non-payment of the agreed amount. Debt holders may, for instance, agree to a partial write-off of their claims; default costs (such as legal fees) may arise; parts of the firm's equity may be transferred to the creditors; or the claims may simply be deferred rather than forgiven.

[26] For details, we refer to Kruschwitz et al. (2015, p. 211f.).

Precisely because the range of possible outcomes is so broad, we chose not to go into these details earlier. Such an analysis would require information about the firm's equity and debt holders that goes far beyond what is typically assumed in the theory of discounted cash flow. Most importantly, one would need to understand the relative bargaining power of both parties and the range of possibilities available to them for renegotiating their contractual arrangements. We will now take a somewhat closer look at this problem.

Once insolvency occurs, the model must specify what happens next. There are several plausible ways to proceed, and we now consider one of them: the transfer of equity in bankruptcy. We do not choose this mechanism because it is necessarily the most realistic one, but because it can be analyzed with the tools developed so far.

To begin with, we consider cases in which, upon insolvency, equity are transferred from the shareholders to the debt holders ("transfer of equity"). This transfer may involve either all of the firm's equity or only a fraction $\alpha \leq 100\%$. We impose the following ordering: cash-flow claims are satisfied first, whereas the firm value, if needed, is transferred only proportionately, with share α,

$$\text{default in state } \omega \text{ at time } t+1$$
$$\Longrightarrow \qquad \widetilde{I}_{t+1}(\omega) + \widetilde{Pr}_{t+1}(\omega) := \widetilde{CF}^{l}_{t+1}(\omega) + \widetilde{D}_{t+1} + \alpha \widetilde{E}^{l}_{t+1}(\omega) \; .$$

Consequences for the Coupon We now turn to the case of a full transfer of equity ($\alpha = 1$) and examine its implications for the two insolvency triggers. To do so, let us combine the relevant terms whose signs determine insolvency,

$$\begin{aligned} &\overbrace{\widetilde{CF}^{l}_{t+1} + \widetilde{D}_{t+1} - (1+c_t)\widetilde{D}_t}^{\text{illiquidity}} + \overbrace{\widetilde{V}^{l}_{t+1} - \widetilde{D}_{t+1}}^{\text{over-indebtedness}} \\ &\qquad = \underbrace{\widetilde{CF}^{l}_{t+1} + \widetilde{V}^{l}_{t+1}}_{\text{total assets}} - \underbrace{(1+c_t)\widetilde{D}_t}_{\text{creditor's claims}} \; . \end{aligned} \tag{2.23}$$

The term on the right-hand side represents the total assets available to creditors, $\widetilde{CF}^{l}_{t+1} + \widetilde{V}^{l}_{t+1}$, minus their contractual non-default claim, $(1+c_t)\widetilde{D}_t$. It thus measures the deviation (positive or negative) from what the creditors would have received in the solvent case.

Insolvency arises when at least one of the two triggers becomes active, meaning that one of the left-hand terms in (2.23) turns negative. Yet this does not necessarily imply that the entire sum is negative. In other words: even if one trigger is violated, this alone does not determine the sign of the total sum; what matters is the magnitude of the deviation of the other trigger.

Suppose insolvency is caused by illiquidity, so that the first term is negative. If the sum nevertheless remains positive, under complete transfer of equity, the creditors would actually gain from the insolvency—that is, receive more than is due to them. Arbitrage requires then that the agreed coupon rate c in the solvent state will be below the risk-free rate r_f.[27] We regard such a situation as unrealistic, since we are not aware of any bonds that require an (effective) interest rate below the risk-free rate.

We therefore distinguish between two cases.

1. If both insolvency triggers are met simultaneously, entire equity must be transferred to the debt holders. In that case, the transfer parameter is $\alpha = 1$.
   Since, in such a situation, future cash flows from that insolvency state onward will no longer accrue to the equity holders, one can no longer assume that the original debt policy will remain in place either. We must therefore proceed on the basis of a different financing policy, and once we do so, the conclusion on page 85 can no longer be maintained: tax effects will then arise.
2. If, by contrast, only one of the two insolvency triggers is met or the sum of the two triggers is positive and the debt holder would receive more then they are entitled to, the simplest way to resolve this conflict is then to set the coupon equal to the risk-free rate and transfer only a fraction of the firm, so that $\alpha < 1$. In the insolvency literature, this case is often captured by requiring that the coupon should never fall below the risk-free rate. We adopt this requirement here.
   If furthermore we continue to maintain the financing policy used so far, the equations derived without taking insolvency into account remain valid.[28]

2.3.3 Examples and Problems

In this section, we extend our numerical examples to include insolvency risk. Once again, we will examine both the finite and the infinite binomial model. As explained above, we now consider one particular case among the possible arrangements in default: either all of the firm's equity or only a fraction of it is transferred to the debt holders.

So far, we have not fully explained how the coupon rate c_t is determined. We have merely stated that debt holders choose the coupon rate such that the loan they grant constitutes a fair deal for them (see Fig. 2.6). The obstacle was the absence of a specific stochastic structure for cash flows and also the lack of a detailed rule describing the resolution of default. In what follows, we consider a binomial model without specifying whether it is finite or infinite. We focus on a single node ω, from which exactly two possible future states may arise one period later: up, denoted by ωu, and down, denoted by ωd.

As in Sect. 2.1.5, we simplify the notation here as well. We are at an arbitrary node ω and in the notation for debt $\widetilde{D}_t$ and for the value of the levered firm $\widetilde{V}_t^l$, we suppress explicit reference to that node. The subsequent firm values in the up and down states are written as $\widetilde{V}_{t+1}^l(u)$ and $\widetilde{V}_{t+1}^l(d)$, respectively, without mentioning the node ω again.

[27] Compare Fig. 2.6.

[28] In the previous edition of our book, we had not yet fully understood this point. Some of the calculations in that edition suffered precisely from this problem, and we have therefore omitted them in the present version.

Accordingly, $Q(u)$ and $Q(d)$ represent the (conditional) risk-neutral probabilities of these two successor states of ω.[29]

We now derive an equation for $c_t(\omega)$, even though it is not yet clear whether insolvency will actually occur. This ordering is necessary, since the coupon rate must be determined before one can check whether an insolvency trigger is met. In particular, illiquidity cannot be examined first, because Definition 2.4 already contains the term c_t.

Coupon Rate in the Binomial Model We consider default only in the down state, since it would make little sense for insolvency to occur in both states—under such conditions, no lending agreement would ever be executed. At state ωu the coupon rate c_t is paid to the creditors. If, however, ωd occurs, the firm becomes insolvent, and its equity are transferred to the creditors. What can be said about the coupon rate c_t?

If default occurs at time $t+1$, we need the amount of debt from the preceding period, $\widetilde{D}_t$, to determine the coupon rate. The debt holders act rationally and therefore apply the Fundamental Theorem of Asset Pricing to assess whether a lending agreement is worthwhile for them. Starting from a partial transfer of equity in the case of default yields

$$\widetilde{D}_t = \frac{(1+c_t)\,\widetilde{D}_t\,Q(u) + \big(\widetilde{CF}^l_{t+1}(d) + \widetilde{D}_{t+1}(d) + \alpha\widetilde{E}^l_{t+1}(d)\big)\,Q(d)}{1+r_f}\;.$$

We will use this approach to determine the coupon rate explicitly. But this equation contains an additional degree of freedom, namely α, whose value is not yet determined. We therefore proceed in two steps. First, we set $\alpha = 1$ and examine the coupon implied by a complete transfer of equity. If the resulting coupon is not smaller than the risk-free rate, the case is internally consistent and the transfer of equity to the debt holders must be complete. If, however, the implied coupon falls below the risk-free rate, a complete transfer would overcompensate the creditors. In that case, we impose $c = r_f$ and determine α so that only a partial transfer takes place.

Thus, the relation above gives rise not to one, but to two equations, one for the coupon c and one for the fraction α; in the first equation, some simplifications have already been made:

$$(1+r_f)\widetilde{D}_t = \widetilde{CF}^l_{t+1}(d) + \widetilde{D}_{t+1}(d) + \alpha\widetilde{E}^l_{t+1}(d) \tag{2.24}$$

$$(1+r_f)\widetilde{D}_t = (1+c_t)\,\widetilde{D}_t\,Q(u) + \big(\widetilde{CF}^l_{t+1}(d) + \widetilde{V}^l_{t+1}(d)\big)\,Q(d)\;. \tag{2.25}$$

2.3.3.1 The Finite Example (Continued)

Let us turn to our finite example. To simplify matters, we disregard taxes. We assume that the provisional leverage policy takes the form

$$D_0 = 140, \quad D_1 = 62, \quad D_2 = 0\;,$$

[29] A more complicated notation for conditional probability was introduced in Eq. (1.6).

and ask whether this particular financing policy can lead to bankruptcy.

We first compute the cash flows accruing to equity holders. Starting from the firm's cash flows,[30] we subtract interest and principal repayments using the coupon rate $c_0 = r_f$. This yields, in each state, the corresponding amounts $\widetilde{CF}_t^u - (1 + r_f)\widetilde{D}_{t-1} + \widetilde{D}_t$ in Fig. 2.7. The illustration shows that at $t = 1, 2$ insolvency (in terms of interruption of payments or illiquidity) cannot occur. Hence, in these states, the financiers will agree on a nominal rate of $c_t(\omega) = r_f$, because all creditors' claims can be satisfied. Time $t = 0$ is different. While the creditors' claims can be met in the up state, this is not possible in the down state.[31]

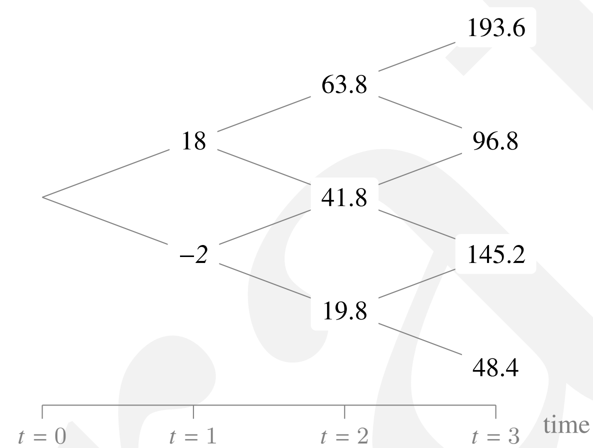


Fig. 2.7 Shareholder's claims in the finite example in Sect. 2.3.3.1 with a coupon of $c = r_f$ (if insolvency is ignored).

Hence, insolvency will occur at $t = 1$ in the down state. We begin by evaluating the coupon rate c_0 and refer to Eq. (2.25) adapted to our case where a complete transfer of equity takes place:[32]

$$140 \approx \frac{(1 + c_0) \times 140 \times 0.0833 + (90 + 158.13) \times 0.9167}{1 + 10\%} \qquad \Longrightarrow \qquad c_0 \approx -729.9\% \,.$$

This example nicely illustrates the discussion in the preceding section. We have noted that when only one insolvency trigger is active, it may occur that, under a full transfer of equity in the event of default, the creditors are overcompensated. This becomes apparent from a coupon rate that is even below zero. We made a mistake by not recognizing that a complete transfer would repay creditors many times the amount of their loan.

[30] These cash flows can be found in Fig. 1.2.

[31] Note, however, that the liquidity shortfall could in principle be eliminated by extending a loan to $D_2 = 64$ at date $t = 2$: in that case, equity holders would receive a cash flow of zero. However, we stipulated that debt policy is given and not itself the subject of change during calculations.

[32] The firm values were calculated in Sect. 1.4.6, and the probabilities are described in Fig. 2.3.

A partial transfer of equity can restore a more realistic outcome. But we must be careful not to change the debt policy at this point. This restriction is one we imposed ourselves: by assumption, the financing policy is fixed and is not revised ex post once insolvency occurs. For that reason, we do not pursue alternative arrangements, since doing so would require more detailed information about the creditors, information that goes beyond the DCF model itself. Instead, we continue within the framework of our own assumptions and consider what happens if the missing cash flow of 2 is provided through a partial transfer of ownership. Other arrangements would of course be conceivable, but pursuing them would require a change in financing policy and would thereby make our setup inconsistent.

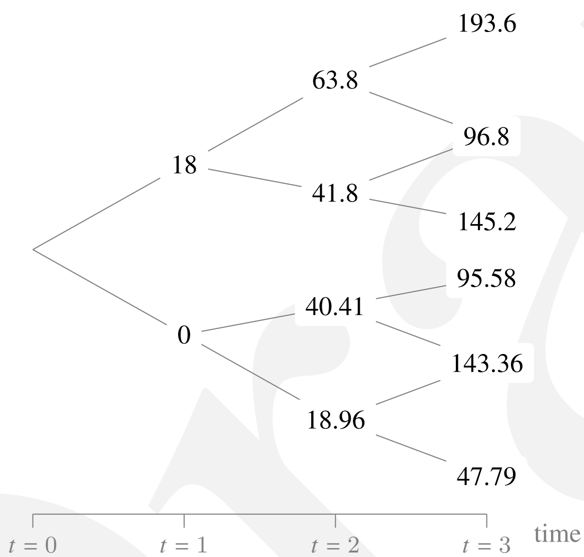


Fig. 2.8 Shareholder's claims in the finite example in Sect. 2.3.3.1 with a coupon of $c = r_f$ (if insolvency is taken into account and a partial transfer of $\alpha \approx 1.26\%$ takes place).

The fraction of the firm that creditors can claim from shareholders is easy to determine using Eq. (2.24):

$$\alpha = \frac{2}{158.126} \approx 1.26\% \,.$$

In the solvent states (following an up move at $t = 1$), the payments are given by the contractual debt schedule $\widetilde{CF}_t^u - (1 + r_f)\widetilde{D}_{t-1} + \widetilde{D}_t$.

Figure 2.8 reports the cash flows accruing to shareholders once default is taken into account. We find that, under this arrangement, no insolvency trigger is activated any longer.

This example reveals how default is handled in such a situation. Since we did not want to alter the firm's total cash flows, insolvency changed only the distribution between equity holders and debt holders. Had we included taxes, the tax advantages would likewise have remained unchanged even in the presence of insolvency. One might summarize the example as follows: insolvency operates here as if later cash flows were "brought forward" and paid earlier to the equity holders, thereby avoiding insolvency. Nothing else changes. With this remark, we conclude the discussion of the finite example.

2.3.3.2 The Infinite Case (Continued)

We turn to insolvency in the infinite-horizon case. Unlike in the preceding example, however, we consider a setting in which illiquidity may arise at multiple points in time: starting from ω at time t, the firm continues along the subsequent up states but becomes bankrupt in any down state. Figure 2.9 illustrates the situation we have in mind.

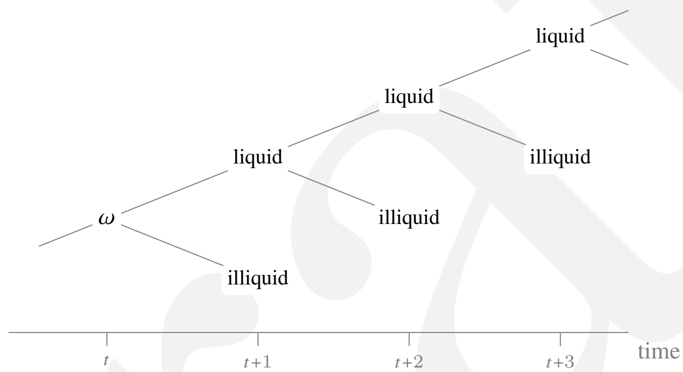


Fig. 2.9 The infinite example with insolvency in each down state.

We need to fully specify the financing policy underlying the illiquidity condition described above. This comes down to determining the debt level. At this point, we are getting slightly ahead of the exposition and assume a constant leverage ratio $l = \frac{\widetilde{D}_t}{\widetilde{V}_t^l}$. This financing policy will be examined in greater detail later, in the section on market-value-based financing.[33]

We do not consider over-indebtedness here, since in this case the condition would be straightforward: one simply chooses a leverage ratio $\frac{\widetilde{D}_t}{\widetilde{V}_t^l}$ above one. Only illiquidity will trigger bankruptcy.

As in the finite example, we disregard taxes in order to keep the analysis simple. Hence, the levered as well as the unlevered firm values coincide: $\widetilde{V}^l = \widetilde{V}^u$.[34]

It suffices to restrict attention to a single period, since from the chosen time t onward the same structure repeats. Hence, we consider the state ω at time t and its two successor states, u and d.

[33] See Sect. 2.5.

[34] This section poses a typographical challenge. Both the up movement and the unlevered firm are denoted by the same letter u. Since this ambiguity arises only in this section, we keep the existing notation unchanged throughout the rest of the section.

As above, we simplify the notation again. We are at a node $\omega = u \dots u$ and in the notation for debt $\widetilde{D}_t$ and for the value of the levered firm $\widetilde{V}_t^l = \widetilde{V}_t^u$, we suppress explicit reference to that node. The subsequent firm values in the up and down states are written as $\widetilde{V}_{t+1}^l(u) = \widetilde{V}_{t+1}^u(u)$ and $\widetilde{V}_{t+1}^l(d) = \widetilde{V}_{t+1}^u(d)$, respectively, without mentioning the node ω again.

Insolvency trigger We are interested in the leverage ratio l at which illiquidity arises. Naturally, this will occur in the down state. At the critical leverage ratio at which illiquidity first emerges, that is, in the boundary case, the coupon will still be set at $c = r_f$, so that the relevant condition can be written as follows:

$$\begin{aligned} 0 &> \widetilde{CF}_{t+1}^l(d) - \left((1+r_f)\widetilde{D}_t - \widetilde{D}_{t+1}(d)\right) && \text{by (2.21)} \\ &= k^{E,u}\widetilde{V}_{t+1}^u(d) - \left((1+r_f)l\widetilde{V}_t^u - l\widetilde{V}_{t+1}^u(d)\right) && \text{Gordon-Shapiro} \\ &= dk^{E,u}\widetilde{V}_t^u - \left((1+r_f)l\widetilde{V}_t^u - ld\widetilde{V}_t^u\right) && \text{down movement} \end{aligned}$$

and this inequality is satisfied if

$$l > \frac{dk^{E,u}}{1 + r_f - d} =: l_1 \ . \tag{2.26}$$

If the leverage ratio l exceeds the threshold l_1, the firm will be illiquid in every down movement.

(Preliminary) Coupon rate In the next step, we now examine when the coupon exceeds the risk-free rate. To do so, we determine the coupon at $l = l_1$ in the case of a complete transfer of equity. We use (2.25) and since taxes were disregarded in this section, this simplifies to

$$l_1(1+r_f) = \left(1 + c_t(l_1)\right) l_1\, Q(u) + d(1 + k^{E,u})\, Q(d) \ .$$

The coupon $c(l_1)$ does not depend on t and we obtain

$$c(l_1) = r_f + \frac{1 + r_f - u(1 + k^{E,u})}{k^{E,u}} \ . \tag{2.27}$$

It can be verified that, under our assumptions, this value is always *below* the risk-free rate.[35]

We already know what this implies: At the threshold l_1, a complete transfer of equity would imply that debt holders receive too much in the liquid state. In such a situation, one would therefore agree on only a partial transfer of equity, and the coupon would then be equal to r_f.

We define another threshold

$$l_2 := \frac{1 + k^{E,u}}{1 + r_f} d \ .$$

[35] See Prob. 2.11. One might ask how far the coupon rate deviates from the risk-free rate. Exercise 2.10 addresses this question and shows that the difference can, in fact, be quite substantial.

A simple calculation shows that $l_1 < l_2$ follows from (2.7). Furthermore, using Eq. (2.25) we can verify that for $l > l_2$ a complete transfer of equity is necessary to satisfy the debt holders' claims.

With the help of these intermediate remarks, we can finally summarize. There are three cases:

$l \leq l_1$ No insolvency occurs.

$l_1 < l < l_2$ The firm is illiquid. However, bankruptcy can be avoided through a partial transfer of equity. As a result, the owners hold a progressively smaller share of the firm over time. In each period, the coupon is $c = r_f$.
In this case, the amount of α given by Eq. (2.24) describes the share that the equity holders must transfer

$$l\,(1 + r_f) = (1 + r_f)\, l\, Q(u) + d(l + \alpha(1 - l) + k^{E,u})\, Q(d)$$
$$\Longrightarrow \quad \alpha = \frac{l - l_1}{l_2 - l_1}\,\frac{1 - l_2}{1 - l}\ .$$

$l_2 \leq l$ The firm is illiquid, and a complete transfer of equity occurs. The coupon c rises above the riskless rate and is given by (2.25).

This example illustrates a point that earlier arose only in abstract form, namely how insolvency can affect valuation. Here, insolvency brought about a partial transfer of ownership, with debt holders taking over part of the firm. Once this is taken into account, the financing policy is no longer the same as the one originally postulated, since part of the debt is effectively extinguished. This, in turn, changes interest payments and, if taxes were included, any resulting tax advantages as well. The value of the levered firm in the presence of insolvency would therefore be lower than before, but only to the extent of the tax shield, since wealth is transferred from debt holders to equity holders.

2.3.3.3 Problems

Problem 2.9 If one wishes to examine the definition of illiquidity, the coupon rate must first be determined. In this exercise, we show under which conditions the risk-free rate may be used instead.

Consider a binomial model and demonstrate, based on (2.25): if the firm is not over-indebted and if it is known that $c > r_f$, then illiquidity already follows from

$$\widetilde{CF}^{u}_{t+1}(\omega d) - (1 + r_f)\widetilde{D}_t(\omega) + \widetilde{D}_{t+1}(\omega d) < 0$$

where r_f replaced c in (2.21).

Problem 2.10 Consider the infinite-horizon case with insolvency risk where $u = 1.1$, $d = 0.7$, $r_f = 10\%$ and $k^{E,u} = 20\%$. Show that $c = -100\%$ at the threshold $l = l_1$ (assuming a complete transfer of equity). Determine l_2 and α as a function of l.

Problem 2.11 Verify Eq. (2.27) and show that c is below the risk free rate.

2.4 Autonomous Financing

2.4.1 Adjusted Present Value (APV)

When all future amounts of debt are already determined at valuation date $t = 0$, we speak of autonomous financing. In this case, the firm follows a fixed redemption plan. We are not concerned with whether autonomous financing is a realistic assumption; however, we suppose it is reasonably close to reality.

Definition 2.5 (Autonomous financing) *A firm is autonomously financed if and only if its future debt amounts $\widetilde{D}_t$ are deterministic today.*

Under this assumption, the tax advantages are deterministic and can be discounted at the risk-free rate. This leads straight to the so-called APV formula. The abbreviation stands for adjusted present value.

Theorem 2.6 (APV formula) *In the case of autonomous financing, the following equation is valid for the market value of the levered firm*

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \sum_{s=t+1}^{T} \frac{\tau r_f D_{s-1}}{\left(1+r_f\right)^{s-t}} \ .$$

This theorem can be proved as follows. We start from Eq. (2.14) and use that our firm follows a fixed redemption policy. If, however, the $\widetilde{D}_t$ are no longer random variables, then the last equation simplifies, by Rule 3 (Certainty), to

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{\tau r_f D_t}{1+r_f} + \ldots + \frac{\tau r_f D_{T-1}}{\left(1+r_f\right)^{T-t}} \ .$$

That finishes the proof. □

Notice that our equation also holds for a firm whose debt is threatened by insolvency. The value of a firm in danger of bankruptcy corresponds exactly to the value above, as long as the granted debt is fixed. Although bankruptcy may occur, firm value is still determined by discounting the tax advantages from the granted debt at the risk-free interest rate, not at the cost of debt or any other discount rate. What appears at first glance to be an unexpected result arises because creditors anticipate the threat of bankruptcy and therefore require interest payments that exactly compensate any loss in value due to insolvency.

Long-Term and Constant Amount of Debt Finally, we want to look at the case where the amount of debt always remains the same. We then require that at time t the current debt will not change through redemption or further loans. In this case the last valuation formula can be simplified to an equation, which is named after those who discovered it.

Theorem 2.7 (Modigliani-Miller formula) *If the firm goes on forever, the conditions of Thm. 2.6 are met and debt remains constant, then the following is valid for the market value of the levered firm*

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \tau D_t \ .$$

To prove the theorem we convert the right summands in Thm. 2.6 with help of the geometric annuity. If debt remains constant we get

$$\begin{aligned}\widetilde{V}_t^l &= \widetilde{V}_t^u + \tau r_f D_t \sum_{s=t+1}^{\infty} \frac{1}{(1+r_f)^{s-t}} \\ &= \widetilde{V}_t^u + \tau r_f D_t \frac{1}{r_f} \ .\end{aligned}$$

And that is what we wanted to show.[36] □

We obtain a slightly different representation if we focus on time $t = 0$ and assume that the expected cash flows are constant. The amount of debt D_0 as the product of the debt ratio l_0 and market value of the levered firm can be written,

$$\begin{aligned}V_0^l &= V_0^u + \tau\, l_0\, V_0^l \\ (1-\tau\, l_0)V_0^l &= V_0^u \ .\end{aligned}$$

If we use the assumption that the expected free cash flows remain the same in time, then we can calculate the market value of the unlevered firm as cash value of a perpetual return, from which

$$(1-\tau\, l_0)\, V_0^l = \frac{\mathrm{E}\left[\widetilde{CF}^u\right]}{k^{E,u}}$$

and

$$V_0^l = \frac{\mathrm{E}\left[\widetilde{CF}^u\right]}{(1-\tau\, l_0)k^{E,u}}$$

follow. The last equation is also called Modigliani-Miller adjustment. The valuation equation proven in this chapter solves the problem as to how the market value is to be

[36] Notice that transversality for debt is trivially satisfied in this case since $\widetilde{D}_t$ is constant.

determined in full. We will come back to this equation once more in relation to the Miles-Ezzell adjustment.[37]

2.4.2 Examples and Problems

2.4.2.1 The Finite Case (Continued)

Here debt will be risk-free, hence $\widetilde{k}_t^D = r_f$. If we suppose a future development of the amount of debt of

$$D_0 = 100, \quad D_1 = 100, \quad D_2 = 50\ ,$$

the value of the levered firm at time $t = 0$ is gotten with

$$\begin{aligned} V_0^l &= V_0^u + \frac{\tau r_f D_0}{1+r_f} + \frac{\tau r_f D_1}{\left(1+r_f\right)^2} + \frac{\tau r_f D_2}{\left(1+r_f\right)^3} \\ &\approx 229.75 + \frac{0.5 \times 0.1 \times 100}{1.1} + \frac{0.5 \times 0.1 \times 100}{1.1^2} + \frac{0.5 \times 0.1 \times 50}{1.1^3} \approx 240.30\ . \end{aligned}$$

The value of the levered firm is not materially different from that of the unlevered firm. This is because, in our example, the firm has a lifetime of only three years.

We deduce the market value of the levered firm at time $t = 1$ in an analogous way,

$$\begin{aligned} \widetilde{V}_1^l &= \widetilde{V}_1^u + \frac{\tau r_f D_1}{1+r_f} + \frac{\tau r_f D_2}{\left(1+r_f\right)^2} \\ &= \widetilde{V}_1^u + \frac{0.5 \times 0.1 \times 100}{1.1} + \frac{0.5 \times 0.1 \times 50}{1.1^2} \\ &\approx \begin{cases} 199.88, & \text{if the development in } t = 1 \text{ is up,} \\ 164.74, & \text{if the development in } t = 1 \text{ is down.} \end{cases} \end{aligned}$$

Finally, we want to determine another additional outcome of our example. The levered firm's debt ratio at time $t = 1$ is a random variable. The following is valid

$$\widetilde{l}_1 \approx \begin{cases} \frac{100}{199.88} = 50.03\%\ , & \text{if the development in } t = 1 \text{ is up,} \\ \frac{100}{164.74} = 60.70\%\ , & \text{if the development in } t = 1 \text{ is down.} \end{cases}$$

If the firm is following an autonomous financing policy, it cannot be supposed that the debt ratio is deterministic. That is much more characteristic of financing based on market values.

[37] See Sect. 2.5.4.

2.4.2.2 The Infinite Case

Let the tax rate be $\tau = 50\%$. The levered firm maintains a debt schedule such that the amount of debt remains constant, $D_t = 100$. Debt is not threatened by insolvency. As already indicated in the Modigliani-Miller formula the value of the levered firm is given by

$$V_0^l = V_0^u + \sum_{t=0}^{\infty} \frac{\tau r_f D_t}{\left(1 + r_f\right)^{t+1}} = V_0^u + \tau D_0 = 500 + 0.5 \times 100 = 550 \; .$$

2.4.2.3 Problems

Problem 2.12 Let debt be risk-free. Assume that the cash flows of the unlevered firm are martingale-like. What assumptions on the debt schedule are necessary such that the cash flows of the levered firm $\widetilde{CF}_t^l$ are martingale-like as well?

Problem 2.13 Show that in the infinite example with constant (risk-free) debt for all t

$$\widetilde{V}_t^l = \frac{\widetilde{CF}_t^u}{k^{E,u}} + \tau D_t \; .$$

Problem 2.14 Show in the infinite example with constant (risk-free) debt that the dividend-price ratio of the levered firm is a random variable (i.e., not deterministic) if $r_f \neq k^{E,u}$.

2.5 Financing Based on Market Values

Let us turn our attention now to a further financing policy. In the following we will suppose that the managers of the firm to be valued are following a policy based on market values.

Definition 2.6 (Financing based on market values) *A firm's financing is based on market values if its debt ratios* $\widetilde{l}_t$ *are already deterministic quantities today.*

A characterization of this financing in another context was that the firm's leverage "breathes" with the equity's market value. If the equity's market value changes, then the amount of debt has to be adjusted.

Stochastic Tax Advantages If a firm's debt ratios, rather than its amount of debt, are given, then autonomous financing no longer applies. To illustrate the consequences, we suppose that management is aiming towards coming up with a debt ratio of 50:100, for instance, at a future time $t > 0$.[38] If we solve the debt ratio's definition Eq. (2.8) according to $\widetilde{D}_t$, then with a deterministic ratio we get

$$l_t \widetilde{V}_t = \widetilde{D}_t \ .$$

This means: if $\widetilde{V}_t$ is uncertain today, then the debt amount at that time is also stochastic, since multiplying a random variable by a constant yields a random variable. In particular, half of a random variable is still a random variable.

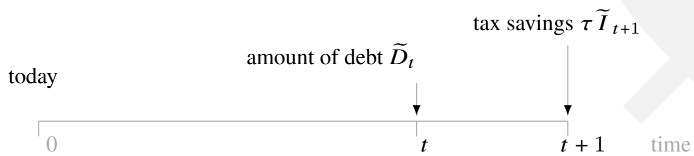


Fig. 2.10 Time structure for market-value-based financing.

In order to fully understand the relations, look at Fig. 2.10. With financing based on market values, the analyst cannot determine at time $t = 0$ the amount of debt at time t, since $\widetilde{V}_t$ is still stochastic and the debt amount is tied to it via the debt ratio. Once this uncertainty is resolved—immediately after time t—the interest payments due at $t+1$ are, by contrast, deterministic (neglecting insolvency). The same holds for the resulting tax savings $\tau \widetilde{I}_{t+1}$ at $t+1$. These tax savings should therefore be discounted at the risk-free rate from $t+1$ back to t. However, discounting from t to 0 cannot be done at r_f: from today's perspective, the tax savings attributable to the debt-financed portion of the firm are stochastic. Even in the absence of insolvency, discounting from t to 0 cannot use r_f, because at $t = 0$ the magnitude of the tax advantages from debt is unknown.

Miles and Ezzell (1980) have examined this case in detail and have determined that the APV equation of Thm. 2.6 is no longer suitable. The tax savings per credit period are stochastic, and such tax savings cannot be discounted at the risk-free rate. [39]

Three Calculation Procedures In the case of such a leverage policy, there are three different calculation procedures, which all lead to the same value of the firm. This value must differ from the value that would be calculated with the APV equation, since it is not based on an autonomous financing policy.

Which of the three calculations should be used by the analyst depends upon, among other things, the state of information that she has on hand. We point out here that the content of the three equations which we have to prove is independent of financing based on market values: a policy based on market values is not absolutely necessary in all cases. Somewhat weaker conditions are instead enough. What is always important is that particular cost of capital is deterministic. To make this relation (which does at first glance seem complicated) understandable, we will first present the three formulations as

[38] We do not want to further pursue where management is getting this target value from.

[39] "Even though the firm might issue risk-free debt, if financing policy is targeted to realized market values, the amount of debt outstanding in future periods is not known with certainty (unless the investment is risk-free) . . . " Miles and Ezzell (1980, p. 721).

generally as possible. At the end we will then show how they interrelate. It is not until this second step that the assumption of financing based on market values (and that free cash flows of the unlevered firm are martingale-like) will be made use of.

2.5.1 Flow to Equity (FTE)

In this section, we work under the assumption that the levered firm's cost of equity is deterministic. To do so, we must define the cost of equity. We begin by determining the payments to which the equity financiers of the levered firm are entitled. The starting point is the free cash flow of the levered firm, $\widetilde{CF}^{l}_{t+1}$.

We have to deduct the payments to the creditors. These are entitled to interest as well as debt repayments at time $t+1$. The interest claims amount to $\widetilde{I}_{t+1}$. We measure principal repayment as the reduction in the outstanding debt balance, i.e., $\widetilde{Pr}_{t+1}$. In total, the creditors are entitled to payments of

$$\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} \ .$$

Subtracting this from the starting amount leaves the residual for the owners (equity value plus dividend) of

$$\widetilde{E}_{t+1} + \widetilde{Div}_{t+1} := \widetilde{E}_{t+1} + \widetilde{CF}^{l}_{t+1} - \widetilde{I}_{t+1} - \widetilde{Pr}_{t+1} \ . \tag{2.28}$$

This makes clear how the cost of equity should be defined.

Definition 2.7 (Cost of equity of the levered firm) *The cost of equity $\widetilde{k}^{E,l}_{t}$ of a levered firm is the conditional expected return*

$$\widetilde{k}^{E,l}_{t} := \frac{\mathrm{E}_t\left[\widetilde{E}_{t+1} + \widetilde{CF}^{l}_{t+1} - \widetilde{I}_{t+1} - \widetilde{Pr}_{t+1}\right]}{\widetilde{E}_t} - 1 \ .$$

The index E with the cost of capital indicates that the cost of equity is being dealt with.

We now assume that this cost of equity is deterministic. Applied to a publicly traded firm, this would require knowing the firm's future conditional expected returns on its shares. We are fully aware that this assumption is very strong, but it would be remiss not to spell out the implications of the assumptions that underlie DCF theory.

The resulting valuation equation is called the flow-to-equity (FTE) or equity approach in the literature. It discounts not the total after-tax cash flows, but only the after-tax cash flows accruing to the firm's owners.

Theorem 2.8 (Flow to equity) *If the cost of equity of the levered firm $k_t^{E,l}$ is deterministic, then the value of equity at time t is*

$$\widetilde{E}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^l - \widetilde{I}_s - \widetilde{Pr}_s\right]}{\left(1+k_t^{E,l}\right)\dots\left(1+k_{s-1}^{E,l}\right)} .$$

We have already employed the necessary method of proof.[40] We can thus spare our readers it here.

2.5.2 Total Cash Flow (TCF)

If it is supposed that instead of the levered firm's cost of equity the firm's average cost of capital is deterministic, then it is the TCF approach and not the FTE approach which is being dealt with.

First the cost of capital which is now relevant has to be defined again. Which payments do the financiers of the levered firm receive? The answer is straightforward. It is of course the free cash flows of the levered firm $\widetilde{CF}_t^l$ that are the subject here. The capital employed by both groups of financiers equals $\widetilde{E}_t + \widetilde{D}_t = \widetilde{V}_t^l$, so the definition of the cost of capital in this case follows directly.

Definition 2.8 (Weighted average cost of capital – type 1) *The weighted average cost of capital $\widetilde{k}_t^{\varnothing}$ of a levered firm is the conditional expected return*

$$\widetilde{k}_t^{\varnothing} := \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^l\right]}{\widetilde{V}_t^l} - 1 .$$

How close to reality is the assumption of deterministic weighted average cost of capital? If we take a firm listed on the stock exchange for instance, then this condition calls for a knowledge of the entire future returns of the firm. A knowledge is needed thus not of the shares, or the (conditional) expectation of the returns on debt, but rather of the weighted average of both random quantities. A totally different assumption is being dealt with here than that condition on which the FTE formulation is based. We will be able to find out somewhat later if and in how far both conditions are compatible.

[40] See Thm. 1.1.

If we want to know which value the levered firm has at time t, we proceed analogously to Thm. 1.1. This case is termed the total cash flow (TCF) formulation, because the entire after-tax expected cash flows are discounted at the weighted average cost of capital.

Theorem 2.9 (Total cash flow) *If the weighted average cost of capital $k_t^{\varnothing}$ is deterministic, then the value of the firm at time t comes to*

$$\widetilde{V}_t^l = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^l\right]}{\left(1+k_t^{\varnothing}\right)\dots\left(1+k_{s-1}^{\varnothing}\right)} \,.$$

A formal proof could be carried out again just as in Sect. 1.4.3, which is why we forgo with presenting it here again.

What can we say about the weighted average cost of capital (of type 1) and the cost of equity of a levered firm? Can we, for instance, ascertain that deterministic cost of equity also results from deterministic average cost of capital? Are then the FTE and TCF formulations compatible with each other? Or do both assumptions mutually exclude each other? Answers to all these questions are gotten by the so called textbook formula.

Theorem 2.10 (TCF textbook formula) *For the type 1 weighted average cost of capital of the firm, the following relation is always valid*

$$\widetilde{k}_t^{\varnothing} = \widetilde{k}_t^{E,l}\left(1-\widetilde{l}_t\right) + \widetilde{k}_t^D\,\widetilde{l}_t \,.$$

It now becomes clear why we speak of the weighted average cost of capital in relation to the TCF formulation: the levered firm's cost of equity is weighted by the equity ratio, and the cost of debt by the debt ratio.

To prove the theorem, we use the definition of the levered firm's cost of equity. And after a few simplifications, we get

$$\left(1+\widetilde{k}_t^{E,l}\right)\widetilde{E}_t = \mathrm{E}_t\left[\widetilde{E}_{t+1} + \widetilde{CF}_{t+1}^l - \widetilde{I}_{t+1} - \widetilde{Pr}_{t+1}\right] \,.$$

Since the firm's total value equals the sum of equity and debt, we can apply Rule 5 (Known Factor) and, using Definitions 2.3, 2.7, and 2.8, obtain the following:

$$\begin{aligned}\left(1+\widetilde{k}_t^{E,l}\right)\widetilde{E}_t + \left(1+\widetilde{k}_t^D\right)\widetilde{D}_t &= \mathrm{E}_t\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^l\right]\\ \widetilde{V}_t^l + \widetilde{k}_t^{E,l}\widetilde{E}_t + \widetilde{k}_t^D\widetilde{D}_t &= \left(1+\widetilde{k}_t^{\varnothing}\right)\widetilde{V}_t^l \,.\end{aligned}$$

After dividing by the firm's market value, we come up with

$$1 + \widetilde{k}_t^{E,l}\,\frac{\widetilde{E}_t}{\widetilde{V}_t^l} + \widetilde{k}_t^{D}\,\frac{\widetilde{D}_t}{\widetilde{V}_t^l} = \widetilde{k}_t^{\varnothing} + 1 \; .$$

Such is the assertion. □

The textbook formula as it appears in Thm. 2.10 is remarkable in several aspects. To begin with, it is striking that it is obviously valid regardless of whether the relevant variables are understood as random variables or as deterministic quantities. Neither the weighted average cost of capital (type 1) nor the cost of equity of the levered firm (both a prerequisite of the FTE and the TCF approach) need to be deterministic in order to prove the textbook formula. A closer look at the textbook formula allows us to make the following determinations:

1. If the levered firm's weighted average cost of capital (type 1), as well as the cost of equity and the cost of debt, are assumed to be deterministic, then the debt ratios must also be deterministic.
   This is a severe limitation that cannot simply be accepted. However, if we work with deterministic debt ratios, the textbook formula shows how the various costs of capital can be converted into one another. TCF and FTE may alternatively be used and inevitably lead to the same result.
2. If, by contrast, future debt ratios are stochastic in the absence of insolvency, then either the weighted average cost of capital (type 1) or the cost of equity of the levered firm must also be stochastic.
   If the weighted average cost of capital (type 1) is deterministic, then the TCF concept must be used; if, by contrast, the cost of equity is deterministic, then the FTE formulation applies, since a stochastic cost of capital cannot be used as a discount rate. It therefore makes little sense to ask whether the TCF and FTE approaches yield the same result. In the presence of stochastic debt ratios, the textbook formula has no practical value.

What further stands out with the textbook formula as it appears in Thm. 2.10 is that it is different from the usual textbook formula as it is given in relation to the weighted cost of capital (WACC) concept. In the former, the cost of debt cannot be reduced to the firm's profit tax rate. We now want to discuss the question as to whether this cost of capital that is found in the literature also makes economic sense or whether it puts forth totally useless quantities.

2.5.3 Weighted Average Cost of Capital (WACC)

Consider Def. 2.8 of the weighted average cost of capital. We now modify a minor detail and denote the resulting quantities by $WACC$ (weighted average cost of capital): the unlevered firm's free cash flows replace those of the levered firm.

Definition 2.9 (Weighted average cost of capital – type 2) *The cost of capital $\widetilde{WACC}$ of a levered firm is the expected return*

$$\widetilde{WACC}_t := \frac{\mathrm{E}_t\left[\widetilde{V}^l_{t+1} + \widetilde{CF}^u_{t+1}\right]}{\widetilde{V}^l_t} - 1 \ .$$

If this cost of capital is deterministic, we can again prove the existence of a valuation equation analogous to our previous approach. Beforehand, however, we want to discuss how realistic it is to assume that such a cost of capital can be observed in the market.

Anyone who seeks to establish the cost of capital in the sense of Def. 2.9 would have to consider a firm that is, on the one hand, levered ($\widetilde{V}^l_t$) yet, on the other, pays free cash flows as if it were unlevered ($\widetilde{CF}^u_t$). This conflates incompatible quantities; accordingly, we may assume that such a cost of capital cannot be known a priori. Any contrary claim would be untenable.

If we just momentarily free ourselves from this observation and the shackles it would enchain us in, then we could still formulate a valuation equation on the basis of this—albeit somewhat odd—definition of cost of capital. It would read as follows:

Theorem 2.11 (Weighted average cost of capital) *If the levered firm's cost of capital $WACC_t$ is deterministic, then the value at time t of the firm financed by debt comes to*

$$\widetilde{V}^l_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}^u_s\right]}{(1 + WACC_t)\dots(1 + WACC_{s-1})} \ .$$

The proof follows along the same lines as before.

Critical readers will ask why we waste so much time on a cost of capital definition about which we have said that it makes no economic sense. The reason is that this cost of capital definition—the direct interpretation of which makes no sense—will prove itself to be extremely useful in the following.

Theorem 2.12 (WACC textbook formula) *For the firm's type 2 weighted average cost of capital, the following relation is always valid*

$$\widetilde{WACC}_t = \widetilde{k}^{E,l}_t \left(1 - \widetilde{l}_t\right) + \widetilde{k}^D_t \,(1-\tau)\,\widetilde{l}_t \ .$$

The WACC textbook formula differs from the TCF textbook formula only in that the interest on debt is reduced by the tax rate. This corresponds to the standard textbook expression found in the literature. Thus, the weighted average cost of capital equals the levered firm's cost of equity, $\widetilde{k}_t^{E,l}$, weighted by the equity ratio, plus the cost of debt—reduced by the corporate tax rate—weighted by the debt ratio. Our calculation shows that this relation holds regardless of whether the cost of equity and the debt ratio are deterministic or stochastic.

In order to prove the theorem we give the Def. 2.9 of the firm's cost of equity using (2.17) in the form

$$\left(1+\widetilde{WACC}_t\right)\widetilde{V}_t^l = \mathrm{E}_t\left[\widetilde{E}_{t+1}+\widetilde{D}_{t+1}+\widetilde{CF}^l_{t+1}-\tau\left(\widetilde{I}_{t+1}+\widetilde{Pr}_{t+1}+\widetilde{D}_{t+1}-\widetilde{D}_t\right)\right] .$$

Applying Definitions 2.3 and 2.7 we get

$$\begin{aligned}\left(1+\widetilde{WACC}_t\right)\widetilde{V}_t^l &= \mathrm{E}_t\Big[\widetilde{E}_{t+1}+\widetilde{CF}^l_{t+1}-\widetilde{I}_{t+1}-\widetilde{Pr}_{t+1}+ \\ &\qquad +(1-\tau)\left(\widetilde{D}_{t+1}+\widetilde{I}_{t+1}+\widetilde{Pr}_{t+1}\right)-\tau\widetilde{D}_t\Big] \\ &= \left(1+k_t^{E,l}\right)\widetilde{E}_t+\left(\left(1+\widetilde{k}_t^D\right)(1-\tau)+\tau\right)\widetilde{D}_t .\end{aligned}$$

Divided by $\widetilde{V}_t^l$ and observing Def. 2.8, we finally get to the following representation

$$\widetilde{WACC}_t = \widetilde{k}_t^{E,l}\left(1-\widetilde{\widetilde{l}}_t\right)+\widetilde{k}_t^D(1-\tau)\widetilde{l}_t .$$

And that is what we wanted to show. □

This textbook formula also allows for the following conclusions.

1. If the weighted average cost of capital as well as the cost of equity and the cost of debt of the levered firm are assumed to be deterministic at time $t=0$, then the debt ratios must also be deterministic. In this case, financing is based on market values, and the FTE and WACC approaches yield identical firm values.
2. If future debt ratios are stochastic, then in the absence of insolvency either the weighted average cost of capital or the cost of equity of the levered firm must also be stochastic. Consequently, one of the two formulations of the theorem cannot be applied for valuation.

Both textbook formulas can easily be combined into a further equation that establishes a relationship between the two adjusted costs of capital, type 1 and type 2. Specifically, it follows immediately that

$$\widetilde{k}_t^{\varnothing} = \widetilde{WACC}_t + \tau\widetilde{k}_t^D\widetilde{l}_t . \tag{2.29}$$

We summarize our results on market-value–based financing in Fig. 2.11. The illustration shows that the three approaches (FTE, TCF, and WACC) yield the same firm value exactly if the debt ratio is deterministic. In all other cases, at most one of the three can be applied.

The attentive reader will notice that in all statements in this and the former section we have not made use of our assumption that the cash flows are martingale-like. We now turn to the adjustment formulas where this assumption will be necessary.

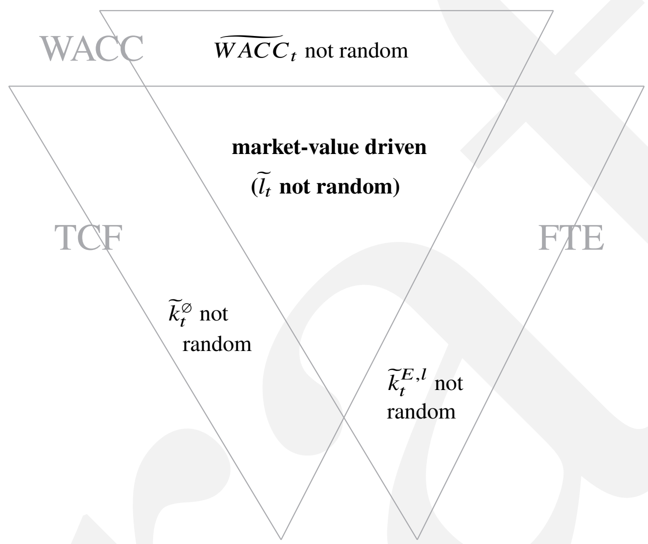


Fig. 2.11 DCF approaches under financing based on market value.

2.5.4 Miles-Ezzell and Modigliani-Miller Adjustments

In the preceding sections, we derived three valuation equations for a levered firm under market-value-based financing: the FTE, TCF, and WACC formulas. We showed that these equations can be used whenever either the levered firm's cost of equity or its weighted average cost of capital (type 1 or type 2) is deterministic. We connected the various costs of capital via the so-called textbook formulas. Market-value-based financing is characterized by deterministic debt ratios. Our analysis of the textbook formulas established that, under this policy, if the cost of capital in one formulation is deterministic, then the corresponding costs of capital in the other two formulations are deterministic as well.

Until now, however, the relationship between these three costs of capital and the cost of capital of an unlevered firm has remained unclear. That is precisely what we consider next. If we assume that the levered firm follows a market-value-based financing policy and, in addition, that the unlevered firm's cost of equity is deterministic, then two questions arise:

1. Under these conditions, are the assumptions of Theorems 2.8 and 2.11 satisfied? In particular, are the levered firm's cost of equity and its weighted average cost of capital also deterministic?
2. Can the levered firm's cost of capital be computed from the unlevered firm's cost of capital?

Adjustment According to Miles and Ezzell The answer to both questions is given by the so called adjustment formula of Miles and Ezzell (1980).

Theorem 2.13 (Miles-Ezzell adjustment formula) *We consider financing based on market values. The unlevered firm's cash flows are martingale-like. If either the unlevered firm's cost of capital or the levered firm's WACC is deterministic, the following relation holds*

$$1 + WACC_t = \left(1 + k_t^{E,u}\right)\left(1 - \frac{\tau r_f}{1 + r_f}\, l_t\right) ,$$

in which all quantities are deterministic.

We prove this theorem as follows. According to the Fundamental Theorem and using Eq. (2.17), the following is valid for the market value of the levered firm

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u + \tau\left(\widetilde{I}_{t+1} + \widetilde{Pr}_{t+1} + \widetilde{D}_{t+1} - \widetilde{D}_t\right)\right]}{1 + r_f} .$$

Since the market value of the firm at time t is already known, the following results from rules 2 and 5 and Eq. (2.20)

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u\right]}{1 + r_f} + \frac{\tau r_f}{1 + r_f}\,\widetilde{D}_t .$$

Using the debt ratio, this can be rewritten in the form

$$\left(1 - \frac{\tau r_f}{1 + r_f}\, l_t\right)\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u\right]}{1 + r_f}$$

or

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u\right]}{\left(1 - \frac{\tau r_f}{1+r_f}\, l_t\right)\left(1 + r_f\right)} .$$

We are thus dealing with a recursive relation from which we have already derived a valuation equation several times.[41] Proceeding in this way, we obtain the following expression for the value of the levered firm,

$$\widetilde{V}_t^l = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^u\right]}{\left(1-\frac{\tau r_f}{1+r_f}\, l_{s-1}\right)\dots\left(1-\frac{\tau r_f}{1+r_f}\, l_t\right)\left(1+r_f\right)^{s-t}}\ .$$

Now we will only get further by falling back upon the Assump. 2.1 and the Thm. 2.3 supported by it. If we use this theorem, we get

$$\widetilde{V}_t^l = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{\left(1-\frac{\tau r_f}{1+r_f}\, l_t\right)\left(1+k_t^{E,u}\right)\dots\left(1-\frac{\tau r_f}{1+r_f}\, l_{s-1}\right)\left(1+k_{s-1}^{E,u}\right)}\ .$$

This yields the recursive relation

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u\right]}{\left(1-\frac{\tau r_f}{1+r_f}\, l_t\right)\left(1+k_t^{E,u}\right)}$$

or

$$\left(1-\frac{\tau r_f}{1+r_f}\, l_t\right)\left(1+k_t^{E,u}\right) = \frac{\mathrm{E}_t\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u\right]}{\widetilde{V}_t^l}\ .$$

A comparison with the Def. 2.9 shows that we have proven the claim. □

If the equation in Thm. 2.13 is combined with the textbook formula from Thm. 2.12, the meaning of the result becomes much clearer. Substituting the textbook formula into the equation from Thm. 2.13 and slightly rearranging yields

$$k_t^{E,l} = k_t^{E,u} + L_t\left(k_t^{E,u} - k_t^D + \tau\left(k_t^D - \frac{1+k_t^{E,u}}{1+r_f}, r_f\right)\right)\ , \qquad (2.30)$$

where L_t denotes the debt–equity ratio as defined in Eq. (2.9). The cost of capital of the levered firm can be determined with this formula if the cost of capital of the unlevered firm, the cost of debt, the income tax rate, the target leverage ratio, and the risk-free interest rate are known. If the above equation is solved for $k_t^{E,u}$—given the leverage ratio and the income tax rate—it provides a way to convert the levered firm's cost of capital into that of the unlevered firm. This requires that the levered firm follow a financing policy based on market values.

The reader may wish to take a moment to revisit Eq. (2.30). It differs in two not-insignificant respects from the original result in Miles and Ezzell (1980). We state these differences explicitly, as the Miles–Ezzell result has become firmly established in today's textbook literature. In their formula, neither the leverage ratio nor the cost of capital

[41] See, for example, our proof of Thm. 1.1 in Sect. 1.4.3.

carries a time index. Accordingly, in the current literature the adjustment formula is not written as in our Thm. 2.13, but rather

$$1 + WACC = \left(1 + k^{E,u}\right)\left(1 - \frac{\tau r_f}{1 + r_f}\, l\right) . \tag{2.31}$$

Miles and Ezzell derived their result under the limitation of the assumption that the cost of capital and the debt ratio are constant in time. As a rule, this limitation is clearly pointed out in the textbook literature. The outcome that we have shown thus has much fewer restrictions than the original result of Miles and Ezzell.

Secondly, our adjustment formula can be applied in case of a firm that might become insolvent. Even though the firm can go bankrupt, still not the cost of debt or any other discount factor but the risk-free interest rate is found in Thm. 2.13. This is a surprising, but an inescapable consequence of our assumptions, that not only the owners, but also the creditors anticipate the threat of bankruptcy.

Problematic Adjustment According to Modigliani and Miller In the discussion of autonomous financing, we mentioned that, in addition to the Miles-Ezzell adjustment, there is another adjustment formula. It is called the Modigliani-Miller adjustment, originates from Theorem 2.7, and takes the form

$$WACC = k^{E,u}(1 - \tau l_0) . \tag{2.32}$$

In practice, this adjustment formula is very popular, supposedly because it looks much more simple than the Miles-Ezzell Eq. (2.31). If you follow the relevant literature, then you get Eq. (2.32) from the conditions of Thm. 2.7 and particularly from the assumption of autonomous financing. Such a firm then cannot be financed based on market values. We will now present a surprising outcome.

Theorem 2.14 (Contradiction of the Modigliani-Miller adjustment) *The cash flows of the unlevered firm are martingale-like. If the weighted average cost of capital of type 2 ($WACC$) and the unlevered firm's cost of equity ($k^{E,u}$) are deterministic, then the firm is financed based on market values.*

First, we will prove the theorem and then consider its significance. To do so, we invoke Theorems 2.1, 2.7, and 2.11 and use that cash flows are martingale-like. The following then holds

$$\left(1-\tau\widetilde{l}_t\right)\overbrace{\sum_{s=t+1}^{T}\frac{(1+g)^{s-t}\,\widetilde{CF}_t^u}{(1+WACC_t)\dots(1+WACC_{s-1})}}^{=\widetilde{V}_t^l}=$$
$$=\underbrace{\sum_{s=t+1}^{T}\frac{(1+g)^{s-t}\,\widetilde{CF}_t^u}{\left(1+k_t^{E,u}\right)\dots\left(1+k_{s-1}^{E,u}\right)}}_{=\widetilde{V}_t^u}.$$

If we shorten $\widetilde{CF}_t^u$, there remains

$$\left(1-\tau\widetilde{l}_t\right)\sum_{s=t+1}^{T}\frac{(1+g)^{s-t}}{(1+WACC_t)\dots(1+WACC_{s-1})}=$$
$$=\sum_{s=t+1}^{T}\frac{(1+g)^{s-t}}{\left(1+k_t^{E,u}\right)\dots\left(1+k_{s-1}^{E,u}\right)}.$$

Besides the debt ratio $\widetilde{l}_t$ we only find deterministic quantities. It is no problem to convert them according to $\widetilde{l}_t$. But then $\widetilde{l}_t$ has to be a deterministic quantity itself. That is what we wanted to show. □

Of what significance is our assertion now? A Modigliani-Miller adjustment formulated as in Eq. (2.32) requires that both the weighted average cost of capital and the unlevered firm's cost of equity be deterministic. The equation does not make sense under other conditions. Under this condition—we have just proved this—the case of financing based on market values is indeed conceivable. And here is where the anomaly lies: the condition of the Modigliani-Miller model is autonomous financing with constant debt. Since a firm cannot be financed both autonomously and based on market values, we have a contradiction. We do not dispute that there are WACC values that, in the case of autonomous financing, yield the correct value of the levered firm. These numbers are appropriate discount rates in the above setup—but they cannot be interpreted as costs of capital in the sense of Def. 2.9. This again highlights that cost of capital and discount rates refer to different economic quantities.

Can we go a step further and claim to have refuted the theory of Modigliani and Miller? Have we perhaps uncovered an error in their reasoning? In answering this question, we must carefully distinguish between two aspects.

In this book, we take the view that the cost of capital represents expected returns. For us, that is the nucleus of a theory of firm valuation. If this line of reasoning is followed, then the ideas of Modigliani and Miller simply do not hold up. Whoever interprets $WACC$ as an expected return cannot simultaneously assume a deterministic debt ratio and the Modigliani-Miller model (with a constant amount of debt) without running into a contradiction.

Approaching the issue from a completely different angle, we might ask whether Modigliani and Miller themselves understood $WACC$ as a cost of capital and were aware of this inconsistency. The answer is unambiguous. For both authors—just as later for Miles and Ezzell—the weighted average cost of capital $WACC$ was conceived merely as a rate that, when used as a discount factor, yields the correct result, namely the correct value of the firm.[42] None of these authors interpreted these quantities as expected returns. This is why we do not mean to imply that Modigliani and Miller were at fault in any way.

2.5.5 Examples and Problems

2.5.5.1 The Finite Case (Continued)

In our example the firm will be financed based on market values. In this case all three approaches FTE, TCF, and WACC will yield the same value of the firm. Since the leverage ratios are given, we concentrate on the WACC approach.

Now we suppose in contrast to the autonomous financing that the following capital structures are realized at future times

$$l_0 = 50\%\,, \quad l_1 = 20\%\,, \quad l_2 = 0\%\,.$$

The weighted average cost of capital results from the Miles-Ezzell equation with

$$\begin{aligned} WACC_0 &= \left(1 + k^{E,u}\right)\left(1 - \frac{\tau r_f}{1 + r_f}\, l_0\right) - 1 \\ &= (1 + 0.2)\left(1 - \frac{0.5 \times 0.1}{1 + 0.1} \times 0.5\right) - 1 \approx 17.27\%\,, \\ WACC_1 &\approx 18.91\%\,, \\ WACC_2 &= 20\%\,. \end{aligned}$$

With that the value of the firm amounts to

$$\begin{aligned} V_0^l &= \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1 + WACC_0} + \frac{\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1 + WACC_0)(1 + WACC_1)} \\ &\quad + \frac{\mathrm{E}\left[\widetilde{CF}_3^u\right]}{(1 + WACC_0)(1 + WACC_1)(1 + WACC_2)} \\ &\approx \frac{100}{1.1727} + \frac{110}{1.1727 \times 1.1891} + \frac{121}{1.1727 \times 1.1891 \times 1.20} \approx 236.46\,. \end{aligned}$$

Financing based on market values obviously leads to a totally different value of the firm than autonomous financing.

We are interested in the value of the company at time $t = 1$. Here we get

[42] That $(1 - \tau l_0) k^{E,u}$ has this character was shown for the case of constant expected free cash flows in Sect. 2.4.1.

$$\widetilde{V}_1^l = \frac{\mathrm{E}_1\left[\widetilde{CF}_2^u\right]}{1+WACC_1} + \frac{\mathrm{E}_1\left[\widetilde{CF}_3^u\right]}{(1+WACC_1)(1+WACC_2)}$$
$$\approx \begin{cases} \frac{121}{1.1891} + \frac{133.1}{1.1891\times 1.20} \approx 195.04, \text{ if up;} \\ \frac{99}{1.1891} + \frac{108.9}{1.1891\times 1.20} \approx 159.58, \text{ if down.} \end{cases}$$

Due to the financing policy based on market values, the payments for the creditors shown in Fig. 2.12 will be yielded in the future. Please note that $l_2 = 0$.[43]

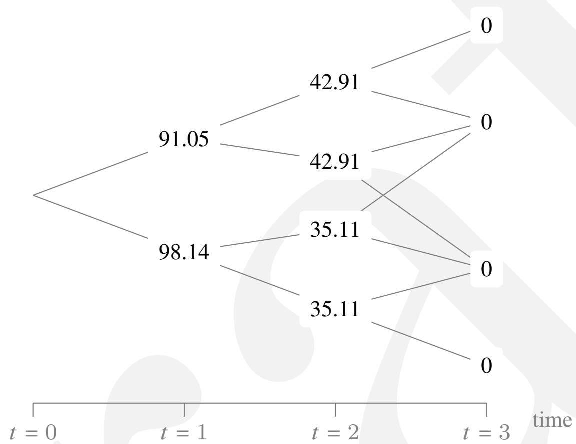


Fig. 2.12 Debtor's claims in the finite example in Sect. 2.5.5.1.

2.5.5.2 The Infinite Case (Continued)

Let us turn to the infinite case. We assume, that the leverage ratio $l = 50\%$ remains constant. If we use the Miles-Ezzell adjustment (Thm. 2.13), the weighted average cost of capital is

$$WACC = (1 + k^{E,u})\left(1 - \frac{\tau r_f}{1 + r_f} l\right) - 1 \approx 17.273\%$$

and from Thm. 2.11 we get

$$V_0^l = \sum_{t=1}^{\infty} \frac{\mathrm{E}_0\left[\widetilde{CF}_t^u\right]}{(1+WACC)^t}$$

[43] Notice that with this financing policy the binomial model is not (fully) recombining: even with a nominal interest rate of 10% the states ud and du do not yield the same cash flows to the debt holder. This is so because the debt and hence the tax shields are different: $\widetilde{D}_2(ud) = l_1 \widetilde{V}_1^l(u) \neq l_1 \widetilde{V}_1^l(d) = \widetilde{D}_2(du)$. The same does not apply at $t = 3$ because $l_2 = 0$.

$$= \sum_{t=1}^{\infty} \frac{CF_0^u}{(1+WACC)^t} = \frac{CF_0^u}{WACC}$$
$$\approx \frac{100}{0.17273} \approx 578.95 \,.$$

This is the value of the levered firm at $t = 0$.

2.5.5.3 Problems

Problem 2.15 Consider a company having (unlevered) cash flows as in Fig. 1.2. The tax rate is 50% and the risk-free interest rate 10%. Assume further that $WACC = 18\%$, but nothing is known about the cost of equity $k^{E,u}$. We do not require that the company maintains a deterministic leverage ratio $\widetilde{l}$ and will show with this problem that the other DCF methods of this chapter are not applicable.

a) Evaluate the value of the firm using the WACC method for $t = 0, 1$.
b) Convince yourself that no assumptions are made yet about the expected future amount of debt since we do not require a deterministic leverage ratio. Instead, for $t = 0, 1$ we will assume the following debt schedule

$$D_0 = 50, \quad \widetilde{D}_1 = \begin{cases} 60 & \text{if up,} \\ 40 & \text{if down.} \end{cases}$$

Determine the leverage ratio at $t = 1$ and show that the firm is not financed based on market values.
c) Evaluate the weighted cost of capital (type 1) at $t = 1$ as well as the cost of equity of the levered firm and show that they both are random variables.

Problem 2.16 Show that for $k^{E,u} > r_f$ the Miles-Ezzell-WACC from (2.31) is always larger than $k^{E,u}(1 - \tau l_0)$ if $l_0 = l$ is the leverage ratio of the firm.

Problem 2.17 Verify that in the infinite example

$$\widetilde{V}_t^l = \frac{\widetilde{V}_t^u}{1 - \frac{1+k^{E,u}}{1+r_f} \frac{r_f}{k^{E,u}} \tau l} \,.$$

(This is the main result of Miles and Ezzell (1985).)

2.6 Financing Based on Book Values

Up to this point, we have considered two forms of financing policy that are frequently discussed in the literature. Under autonomous financing, the borrowing and repayment of debt follow a static, predetermined plan that does not take random developments into account. By contrast, under financing based on market values, the risk of debt is linked to the random development of the equity's market value through a given relation. For exchange-listed firms, this means that when share prices rise, loans are taken out, and when prices fall, debt is repaid. We have repeatedly stressed that we refrain from making claims about which of these financing policies is particularly realistic. If we introduce another financing policy below, it is because we believe it likely plays an important role in business practice. If the managers of a firm announce that they plan to lower or raise the leverage ratio, they are usually measuring that ratio in book values, not market values. These two notions can diverge markedly. This motivates us to analyze a third financing policy in the following section. Like financing based on market values, it is based on debt ratios—this time, however, measured in book values.

2.6.1 Assumptions

With debt and equity as book values, those amounts are in question with which the debts or—according to case—equity are reported in the balance books of the firm to be valued. The first assumption consists in there being no difference between market value and book value of the debt. The credit amounts given in the balance sheets of the firm to be valued correspond to their market prices. Furthermore, we will assume that there is no insolvency risk.

Assumption 2.7 (Book value of debt) *The firm will not become insolvent,*

$$\widetilde{Pr}_{t+1} = \widetilde{D}_t - \widetilde{D}_{t+1} \quad \text{and} \quad \widetilde{I}_{t+1} = r_f \widetilde{D}_t \ .$$

Furthermore, the debt's market value continuously corresponds to its book value,

$$\underline{\widetilde{D}}_t = \widetilde{D}_t \ .$$

It is different with equity. If you start from an initial value $\underline{\widetilde{E}}_t$ and inquire about the book value of equity at time $t+1$, then you have to look at the quantities that can change it:

1. Equity grows if the owners subscribe additional equity as part of a capital increase. The amount of the capital increase between times t and s is denoted by $\underline{\widetilde{e}}^l_{t,s}$.[44]
2. Equity grows further if the managers retain earnings. The levered firm's earnings after taxes at time $t+1$ amount to $\left(\widetilde{EBIT}_{t+1} - \widetilde{I}_{t+1}\right)(1-\tau)$.

[44] It is possible that the capital increase differs between the unlevered and levered companies. At first glance, this may seem paradoxical, because if the two capital increases differ one might expect a statement about the use of the additional funds. Nevertheless, under our assumptions regarding investment and payout policies, no such specification is required. Our model imposes no structure on the asset side of the balance sheet.

3. Equity decreases if the firm pays dividends to the owners. In relation to the FTE approach and using Eq. (2.17), we made clear that payments in the amount of

$$\widetilde{CF}^{u}_{t+1} + \tau \widetilde{I}_{t+1} - \left(\widetilde{I}_{t+1} + \widetilde{D}_t - \widetilde{D}_{t+1}\right)$$

are being dealt with.[45]

For the book value of the equity at time $t+1$, we get a total of

$$\begin{aligned}\underline{\widetilde{E}}^{l}_{t+1} &= \underline{\widetilde{E}}^{l}_{t} + \underline{\widetilde{e}}^{l}_{t,t+1} + \left(\widetilde{EBIT}_{t+1} - \widetilde{I}_{t+1}\right)(1-\tau) \\ &\quad - \left(\widetilde{CF}^{u}_{t+1} + \tau \widetilde{I}_{t+1} - \left(\widetilde{I}_{t+1} + \widetilde{D}_t - \widetilde{D}_{t+1}\right)\right) \\ &= \underline{\widetilde{E}}^{l}_{t} + \underline{\widetilde{e}}^{l}_{t,t+1} + \widetilde{EBIT}_{t+1}(1-\tau) - \widetilde{CF}^{u}_{t+1} + \underline{\widetilde{D}}_t - \underline{\widetilde{D}}_{t+1} .\end{aligned}$$

This gives nothing more than the clean surplus relation.

Assumption 2.8 (Clean surplus relation) *The book value of a levered firm results from*

$$\underline{\widetilde{E}}^{l}_{t+1} = \underline{\widetilde{E}}^{l}_{t} + \underline{\widetilde{e}}^{l}_{t,t+1} + \widetilde{EBIT}_{t+1}(1-\tau) - \widetilde{CF}^{u}_{t+1} + \underline{\widetilde{D}}_t - \underline{\widetilde{D}}_{t+1} .$$

Since $\underline{\widetilde{V}}_t = \underline{\widetilde{E}}^{l}_{t} + \underline{\widetilde{D}}_t$, the book value of the firm's value then obeys the following equation.

Theorem 2.15 (Operating assets relation) *The book value of a levered firm results from*

$$\underline{\widetilde{V}}^{l}_{t+1} = \underline{\widetilde{V}}^{l}_{t} + \underline{\widetilde{e}}^{l}_{t,t+1} + \widetilde{EBIT}_{t+1}(1-\tau) - \widetilde{CF}^{u}_{t+1} .$$

Furthermore, the book values of the levered and the unlevered firm differ only if there are different changes in subscribed capital.

Since earnings before interest and taxes and the free cash flows of the unlevered firm are random variables, the book value of the firm must be stochastic as well.

The financing policy to be looked at in this section is now characterized by the following definition.

[45] See Sect. 2.5.1.

Definition 2.10 (Financing based on book values) *A firm is financed based on book values if the debt ratios to book values* $\underline{\widetilde{l}}_t$ *are deterministic.*

How is a firm's market value to be determined if we suppose a financing policy based on book values? In order to get any further, we have to characterize such a firm more in detail. In doing so we want to differentiate three cases that deal with the investment policy of the firm.

In two of them we have to work with assumptions which combine the volume of investment functionally with other economic ratios of the company. This might surprise the attentive reader due to the principle we assumed earlier. This stated that investments will be carried out only if their net present value is positive. If we adhere to this principle, it is not at all apparent that the volume of investment will be directly linked to specific characteristics like the cash flow or depreciation in a certain way.[46] But admittedly, we are not able to develop valuation equations without such relationships.

1. The firm could carry out a policy of full distribution. In this case the owners annually receive a dividend in amount of the net profit after taxes.
2. The firm could dispense with the policy of full distribution, but carry out an investment policy, which is heavily linked to accruals. In this case the firm would limit itself to undertaking replacement investments.
3. The firm could not link its investments to the accruals but instead to the cash flows. If the cash flows grow, a lot is invested; with low cash flow, in contrast, the investments is scaled back.

We can give valuation formulas for each of these cases. Each of these valuation equations does indeed require that the changes in the subscribed capital are deterministic. Thus

Assumption 2.9 (Subscribed capital) *The changes in subscribed capital* $\underline{\widetilde{e}}^{l}_{t,t+1}$ *are deterministic for all* $t \geq 0$.

Investment and Accruals In the following we will examine three different investment or distribution policies, as the case may be. Although to do so, we need to more precisely define our notion of investment expenses within the framework of our model. For that we look at Fig. 2.4. This presents a relation between the free cash flow, the gross cash flow as well as the investments. Therefore, using (2.11) the following relation is valid for the unlevered firm

[46] There are plenty of research studies about the question of whether the volume of investment is linked to the cash flow of the company, for example. We refer to Fazzari et al. (1988) who show that with market imperfections some firms are constrained in their ability to raise funds externally, hence fluctuations in cash flows account for economically important movements in investment.

$$\widetilde{CF}^u_{t+1} = \widetilde{GCF}_{t+1} - \tau \widetilde{EBIT}_{t+1} - \widetilde{Inv}_{t+1} \ . \tag{2.33}$$

We must also define more precisely the relationship between gross cash flows and accruals within the firm. For that we again take a look at the figure and infer the relation

$$\widetilde{GCF}_{t+1} = \widetilde{EBIT}_{t+1} + \widetilde{Accr}_{t+1}$$

with which the above equation can be simplified to

$$\widetilde{EBIT}_{t+1}(1-\tau) - \widetilde{CF}^u_{t+1} = \widetilde{Inv}_{t+1} - \widetilde{Accr}_{t+1} \ . \tag{2.34}$$

Let us next turn to the full distribution policy.

2.6.2 Full Distribution Policy

It is doubtful whether there are firms that continuously distribute their earnings in full. Analysts in the Anglo-Saxon finance tradition would even regard such a dividend policy as rather foolish. If one believes that managers should undertake all investment projects with positive NPV, then only those funds that cannot earn a profit within the firm would be distributed. If the firm adheres to this idea, the dividend policy is a mere residual. And if managers do not retain earnings as a matter of principle, there is hardly room for far-reaching managerial views about a sensible dividend policy. We nevertheless discuss this case because it has a long tradition in German valuation practice.[47]

The profit after interest and taxes that the levered firm attains amounts to

$$\left(\widetilde{EBIT}_{t+1} - \widetilde{I}_{t+1}\right)(1-\tau) \ .$$

When we were previously discussing the FTE formulation, we already made it clear that the owners of the levered firm annually receive distributions in the amount of (see (2.28))

$$\widetilde{Div}_{t+1} = \widetilde{CF}^u_{t+1} + \tau \widetilde{I}_{t+1} - \left(\widetilde{D}_t + \widetilde{I}_{t+1} - \widetilde{D}_{t+1}\right) \ .$$

In the case of full distribution of the profit made, both amounts must be identical. This leads us to the following definition.

Definition 2.11 (Full distribution) *The levered firm for which*

$$\left(\widetilde{EBIT}_{t+1} - \widetilde{I}_{t+1}\right)(1-\tau) = \widetilde{Div}_{t+1}$$

is valid each time $t \geq 0$*, is following a policy of full distribution.*

[47] See, for example Institut der Wirtschaftsprüfer in Deutschland (2013).

This can be rearranged to

$$\widetilde{CF}^u_{t+1} = \widetilde{EBIT}_{t+1}(1-\tau) + \underline{\widetilde{D}}_t - \underline{\widetilde{D}}_{t+1} \ .$$

Let us make use again of the considerations of the previous section, particularly in relation to gross and free cash flows in Eq. (2.34). We want to use this equation to help us characterize the implicit basis of investment policy. To do so we concentrate on those investments, which are not replacement investments: these are those investments which will be concluded in excess of the difference $\widetilde{Inv}_{t+1} - \widetilde{Accr}_{t+1}$. Eq. (2.34) shows that a full distribution is considered if these investments are financed by debt, and thus if

$$\widetilde{Inv}_{t+1} - \widetilde{Accr}_{t+1} = \left(\underline{\widetilde{D}}_t - \underline{\widetilde{D}}_{t+1}\right)$$

is valid.

The full distribution results in that the equity's book value can change solely due to changes in the subscribed capital: if we enter in the condition of full distribution in the valuation equation (Thm. 2.15), then under Assump. 2.9 this results in

$$\begin{aligned}\underline{\widetilde{V}}^l_{t+1} &= \underline{\widetilde{V}}^l_t + \underline{e}^l_{t,t+1} - \underline{\widetilde{D}}_t + \underline{\widetilde{D}}_{t+1} \\ \underline{\widetilde{E}}^l_{t+1} &= \underline{\widetilde{E}}^l_t + \underline{e}^l_{t,t+1} \ ,\end{aligned}$$

which agrees with what we have stated. Apart from changes to the subscribed capital, the equity's book value remains constant through time.

With no further work, we can now assume that the equity's book value at time $t = 0$ is known. Thus, no random variable is being represented. From this we get

$$\begin{aligned}\underline{\widetilde{E}}^l_{t+1} &= \underline{E}^l_0 + \underline{e}^l_{0,1} + \ldots + \underline{e}^l_{t,t+1} \\ &= \underline{E}^l_0 + \underline{e}^l_{0,t+1} \ .\end{aligned}$$

for the book value of equity at time $t+1$. Since there are only deterministic quantities on the right hand side, the book value of equity at time $t+1$ must be deterministic.

Let us now make use of the fact that the debt ratio $\underline{l}_t$ measured in book values is deterministic. The leverage ratio $\underline{L}_t$, which must be deterministic as well, can be deduced from the debt ratio with no further work. According to definition,

$$\underline{\widetilde{D}}_{t+1} = \underline{L}_{t+1}\left(\underline{E}^l_0 + \underline{e}^l_{0,t+1}\right) \ ,$$

is valid for the book value of debt at time $t+1$, and from that it follows that the book value of debt is deterministic. If we bring this together with Assump. 2.7, we can determine that the market value of debt is deterministic. The firm is autonomously financed. These realizations can be summed up in the following theorem.

Theorem 2.16 (Market value with full distribution) *If a firm is financed based on book values and simultaneously carries out a policy of full distribution, then the following equation is valid for the market value of the levered firm at all times,*

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \sum_{s=t+1}^{T} \frac{\tau r_f \underline{L}_{s-1} \left(\underline{E}_0^l + \underline{e}_{0,s-1}^l \right)}{\left(1 + r_f\right)^{s-t}} \ .$$

We need not repeat the proof here.

2.6.3 Replacement Investments

If a firm solely carries out investments in the scope of its accruals, it foregoes expansion investments and only takes on replacements investments. There are a lot of similarities here to the policy of full distribution just discussed. But it is not totally the same, as we will shortly make clear.

Definition 2.12 (Replacement investment) *A levered firm exclusively takes on replacement investments, if it only invests within the scope of accruals in each period,*

$$\widetilde{Inv}_t = \widetilde{Accr}_t \ .$$

for all $t > 0$.

The main consequence of this definition is that the book value of the firm only changes when the subscribed capital changes. Entering in Eq. (2.34) under Assump. 2.12 brings us to

$$\begin{aligned} \widetilde{EBIT}_t (1 - \tau) - \widetilde{CF}_t^u &= \widetilde{Inv}_t - \widetilde{Accr}_t \\ &= 0 \ . \end{aligned} \tag{2.35}$$

That means: If a firm exclusively takes on replacement investments, then there is no more difference between the profit after taxes of the unlevered firm and that amount that the unlevered firm would distribute to its owners. If we enter in this result into the valuation equation for the book value of the value of the firm (Thm. 2.15), then under Assump. 2.9 there remains

$$\underline{\widetilde{V}}_{t+1}^l = \underline{\widetilde{V}}_t^l + \underline{e}_{t,t+1}^l \ ,$$

and we can recognize that the book value of the firm in fact only can change on the basis of changes in subscribed capital. Since we require that the book value of the firm is known at time $t = 0$, and does not represent a random variable, the following applies

$$\underline{\widetilde{V}}^l_{t+1} = \underline{V}^l_0 + \underline{e}^l_{0,1} + \ldots + \underline{e}^l_{t,t+1}$$
$$= \underline{V}^l_0 + \underline{e}^l_{0,t+1} .$$

Since there are only deterministic quantities on the right hand side, the book value of the value of the firm must be deterministic at time $t + 1$.

If we now take advantage of the firm implementing a financing policy based on book values, then the following is valid for the book value of debt at time $t + 1$

$$\underline{\widetilde{D}}_{t+1} = \underline{l}_{t+1} \left(\underline{V}^l_0 + \underline{e}^l_{0,t+1} \right) .$$

It follows from this that this quantity is deterministic. In relation to Assump. 2.7, it again comes down to the realization that we are dealing with a firm where the financing policy is autonomous. Consequently, we have proven the following theorem.

Theorem 2.17 (Market value with replacement investments) *If a firm is financed based on book values and exclusively carries out replacement investments, then the following equation is valid for the levered firm at each time*

$$\widetilde{V}^l_t = \widetilde{V}^u_t + \sum_{s=t+1}^{T} \frac{\tau r_f \underline{l}_{s-1} \left(\underline{V}^l_0 + \underline{e}^l_{0,s-1} \right)}{\left(1 + r_f\right)^{s-t}} .$$

Long-Term Constant Amount of Debt In the section on autonomous financing, we concentrated on an (exacting) special case, which is mentioned in the literature as the Modigliani-Miller equation.[48] We also want to deal here with this case concerning the debt which continuously stays the same.

Theorem 2.18 (Modigliani-Miller formula based on book values) *The firm lives forever and the conditions of Thm. 2.17 are valid. The debt ratio remains constant through time. The following is then valid for the market value of the firm*

$$\widetilde{V}^l_t = \widetilde{V}^u_t + \tau \underline{l} \left(\underline{V}^l_0 + \underline{e}^l_{0,t} \right) .$$

---

[48] See Sect. 2.4.1.

We cannot recognize any essential difference from the original equation by Modigliani and Miller (see Thm. 2.7). The proof of the theorem is virtually trivial. Since the debt ratio as well as the book value of the value of the firm remain constant, the amount of debt is constant. With that the conditions of the Modigliani-Miller equation (Thm. 2.7) are met. And it is just that which must be shown.

2.6.4 Investment Policy Based on Cash Flows

It is not very often that firms exclusively carry out replacement investments, or have a policy of full distribution. They most often follow an investment policy, which is independent of the accruals. They will, for instance, make expansion investments or occasionally let the capacity of the firm shrink. The considerations of the two previous sections do not help us any further in these much more realistic cases. We have to develop new ideas. Let us start out looking at the investment policy and end up with accruals.

Concerning investment policy we want to follow the idea that the managers constantly reinvest an exogenously predetermined percentage of the cash flows. This percentage may be deterministic and already be fixed at time $t = 0$.

Definition 2.13 (Investments based on cash flows) *We define an investment policy as based on cash flows, if the investments for all future times $t > 0$ are a deterministic multiple of the free cash flows of the unlevered firm,*

$$\widetilde{Inv}_t = \alpha_t \, \widetilde{CF}^u_t \ .$$

You could of course get involved in linking the investment policy to the free cash flows of the levered firm, and thus work with the definition $\widetilde{Inv}_t = \alpha_t \, \widetilde{CF}^l_t$. It does not matter in the end whether you are referring to the free cash flows of the unlevered firm or the levered firm. But we regard our procedure as being justified for the following reasons: if we want to value levered firms with their correct cost of capital, the unlevered firm only ever represents a reference point for us. This reference firm should differ neither in its investment policy nor in its accruals from the actual firm to be valued; see Assump. 2.3 and Def. 2.13.

Our assumption could likewise be viewed critically, because the investments $\widetilde{Inv}_t$ were already deducted from the cash flows $\widetilde{CF}^u_t$. It would then practically be a relation, in which consideration is better given to the gross cash flows after taxes

$$\widetilde{Inv}_t = \beta_t \left(\widetilde{GCF}_t - \tau \widetilde{EBT}_t \right) \ .$$

But from Eq. (2.33) there results after a little reformulating for the unlevered firm

$$\widetilde{Inv}_t = \frac{\beta_t}{1-\beta_t}\,\widetilde{CF}^u_t \ .$$

We come to the following conclusion from this: If parameter $\beta \in (0,1)$ is seen as realistic, then α will typically be greater than zero, but not necessarily smaller than one. Furthermore, values of α are conceivable, which exceed the value of one.

We address ourselves to the accruals now. Provided that the congruence principle applies, the sum of accruals equals the sum of investment payments,

$$\sum_t \widetilde{Accr}_t = \sum_t \widetilde{Inv}_t \ .$$

For this expression the past and future sums need to be determined. However, this equation is not sufficient if more precise statements for the value of the company are needed.

In the literature, one differentiates between accruals which are discretionary and non discretionary. Non discretionary accruals have a definite functional correlation with the investment payments. Furthermore they are marked by a certain regularity. In the following, the non discretionary accruals will be only those which result from a direct linear correlation to the amounts invested. This seems to be especially advisable if the accruals consist solely of depreciations and a straight-line depreciation is applied. Obviously, the congruence principle which we just mentioned is valid in this case as well.

Assumption 2.10 (Non discretionary accruals) *Accruals are established from*

$$\widetilde{Accr}_t = \frac{1}{n}\left(\widetilde{Inv}_{t-1} + \ldots + \widetilde{Inv}_{t-n}\right) \ .$$

If we want to value a firm at time $t = 0$, the investment expenses of the previous periods of time $t = -1$ to $t = -(n-1)$ must also be known. It should not be impossible to obtain this information.

The subsequent calculations show that the lack of discretionary accruals is not of critical importance for the development of a valuation equation. It is sufficient that the accruals linearly depend on investments the investment expenses $\widetilde{Inv}_{-1}$ through $\widetilde{Inv}_{t-n}$.

The Def. 2.13 and the Assump. 2.10 now suffice to prove the following theorem.

Theorem 2.19 (Investment policy based on cash flows) *The cash flows of the unlevered firm are martingale-like and the firm follows a financing based on book values. There are only non discretionary accruals and the investment policy is based on cash flows. The following is then valid for the market value of the levered firm*

$$V_0^l = V_0^u + \tau r_f \sum_{t=0}^{T-1} \underline{l}_t \frac{\underline{V}_0^l + \underline{e}_{0,t} - \sum_{s=1-n}^{0} \frac{\min(n+s,t)}{n} Inv_s}{\left(1+r_f\right)^{t+1}}$$
$$+\tau r_f \sum_{t=1}^{T-1} \frac{\alpha_t \operatorname{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t} \left(\frac{\frac{n}{n}\underline{l}_t}{1+r_f} + \frac{\frac{n-1}{n}\underline{l}_{t+1}}{\left(1+r_f\right)^2} + \ldots + \frac{\frac{1}{n}\underline{l}_{n+t-1}}{\left(1+r_f\right)^n} \right)$$

with $\underline{l}_s = 0$ *for* $s \geq T$.

Since this theorem's proof is very involved, we refer interested readers to the appendix.[49] The equation named in the current theorem is only formulated for time $t = 0$, and is nevertheless anything but pleasant to read. It can be generalized with considerable technical effort so that a result for $\widetilde{V}_t^l$ can be obtained. This representation certainly does not give any new insights. That is why we forego presenting it here.

Long-Term Constant Debt Ratios We also revisit how the valuation equation changes under a few simplifying assumptions. To do so we particularly require that there is no increase in subscribed capital and that the parameters α_t and $\underline{l}_t$ remain constant. We further assume that the firm has an infinite life. In contrast to Thm. 2.7, we do not, however, assume that the cash flows have a constant, or constantly growing, expectation. We can then, nevertheless, substantiate the outcome—which at first seems surprising and is by no means obvious—that a valuation formula, which is very similar to the Modigliani-Miller equation, is valid.

Theorem 2.20 (Adapted Modigliani-Miller formula) *The conditions of Thm. 2.19 are valid. The firms exists perpetually. The debt ratio $\underline{l}$ and the investment parameter α are constant. The influence of past investments on the book value can be disregarded. The following is then valid for the market value of the firm*

$$V_0^l = V_0^u \left(1 + \frac{nr_f - 1 + \left(1+r_f\right)^{-n}}{nr_f} \tau \alpha \underline{l}\right) + \tau D_0 \ .$$

The proof is again found in the appendix.[50] Compared with the original equation of Modigliani and Miller, two additional terms arise that are straightforward to handle mathematically with standard techniques. Nevertheless, the expression is not immediately transparent, so we proceed by simplifying it. For low interest rates, it appears that

$$\frac{nr_f - 1 + \left(1+r_f\right)^{-n}}{nr_f} \approx \frac{(n+1)r_f}{2}$$

[49] See Sect. 5.2.

[50] See Sect. 5.3.

is valid.[51] With that a preliminary estimate of the order of magnitude of this term is easily possible.

Adjustment Formulas We also need adjustment formulas in the case of financing policy based on book values. Whoever wants to, for instance, work with the valuation equation of Thm. 2.19, can only do so if the cost of capital of the unlevered firm $k^{E,u}$ is known. We see two ways of obtaining this information if there is no reference firm available that is actually free of debt.

If we know all variables of the valuation equation except for $k^{E,u}$ of a levered reference firm, the cost of capital can be determined by iteration.[52] The fact that we cannot simply solve the valuation equation according to $k^{E,u}$, would then be a cosmetic blemish at best.

2.6.5 Examples and Problems

2.6.5.1 The Finite Case (Continued)

The duration of depreciation is $n = 2$. Under this condition the assumption that investments will be made exclusively at time $t = 1$ and not afterwards is useful,

$$\alpha_1 = 50\% , \quad \alpha_2 = \alpha_3 = 0\% .$$

The book value of the levered firm at time $t = 0$ is

$$\underline{V}_0^l = 150 .$$

For the debt ratios based on book values, we choose just those ratios, which were also used in the example of financing based on market values,

$$\underline{l}_0 = 50\% , \quad \underline{l}_1 = 20\% , \quad \underline{l}_2 = 0\% .$$

For simplicity's sake, we assume that in the previous periods there were no investments and there were no systematic increases in the subscribed capital,

$$Inv_{-1} = Inv_0 = 0, \quad \underline{e}_{0,2} = 0 .$$

[51] With help of a Taylor expansion the following holds

$$\left(1 + r_f\right)^{-n} \approx 1 - nr_f + \frac{n(n+1)}{2} r_f^2$$

and from that immediately results

$$\frac{nr_f - 1 + \left(1 + r_f\right)^{-n}}{nr_f} \approx \frac{n+1}{2} r_f .$$

[52] In doing so, constant investment parameters α and constant debt ratios $\underline{l}$ would presumably be worked with within the framework of a practical application.

If we employ everything in the valuation equation according to Theorem 2.19, we then get

$$\begin{aligned} V_0^l &= V_0^u + \tau r_f \underline{V}_0^l \sum_{t=0}^{2} \frac{\underline{l}_t}{\left(1+r_f\right)^{t+1}} + \tau r_f \frac{\alpha_1 \operatorname{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}} \left(\frac{\underline{l}_1}{1+r_f} + \frac{\frac{1}{2}\underline{l}_2}{\left(1+r_f\right)^2}\right) \\ &\approx 229.75 + 0.5 \times 0.1 \times 150 \times \left(\frac{0.5}{1.1} + \frac{0.2}{1.1^2}\right) + 0.5 \times 0.1 \times \frac{0.5 \times 100}{1.2} \times \frac{0.2}{1.1} \\ &\approx 234.77\ . \end{aligned}$$

2.6.5.2 The Infinite Case (Continued)

As in the above example we assume that the debt ratio measured in book values remains constant. If we use

$$n = 2, \quad \alpha = 50\%\ , \quad \underline{l} = 50\%\ ,$$

then if there are no investments before $t = 0$ and with Thm. 2.20 we arrive at a firm value of

$$\begin{aligned} V_0^l &= V_0^u + \tau D_0 + \frac{n r_f - 1 + \left(1+r_f\right)^{-n}}{n r_f} \tau \alpha \underline{l} V_0^u \\ &= 500 + 0.5 \times 100 + \frac{2 \times 0.1 - 1 + (1+0.1)^{-2}}{2 \times 0.1} 0.5 \times 0.5 \times 0.5 \times 500 \\ &\approx 678.125\ . \end{aligned}$$

2.6.5.3 Problems

Problem 2.18 Assume that the cash flows follow

$$\widetilde{CF}_{t+1}^u = \widetilde{CF}_t^u + \varepsilon_{t+1}\ ,$$

where ε_{t+1} are independent and normally distributed with expectation zero and variance one. The cost of capital $k^{E,u}$ is constant, and the firm follows an investment policy based on cash flows. There were no investments before $t = 0$ and there will be no increases in subscribed capital in the future. Furthermore, α does not depend on t.

a) Determine the distribution of the cash flows $\widetilde{CF}_t^u$.
b) Write down the perpetual rent formula for the value of the unlevered firm $\widetilde{V}_t^u$. How is the value distributed?
c) *(This problem is hard to solve.)* Write down a simple formula for the book value $\underline{\widetilde{V}}_t$. (You might have to look at the proofs. . .) How is the book value distributed?

Hint: Any addition or any difference of two normally distributed random variables is again normally distributed. The expectation of the sum (or the difference) is the sum (or the difference) of the expectations. If both random variables are independent, then furthermore the variance of the sum is the sum of the variances.

Problem 2.19 Often people use the WACC approach and do not distinguish precisely between market and book values. This problem shows what can go wrong in the case of an infinite rent (i.e., constant expected cash flows).

Assume that a firm is infinitely lived, financed by book-value and follows an investment policy based on cash flows. There were no investments before $t = 0$ and there will be no increases in subscribed capital in the future. Furthermore, α as well as $\underline{l}$ do not depend on t.

Compare the value of the unlevered firm financed by book-value and market-value. Let $\mathrm{E}[\widetilde{CF}^u] = 100$, $r_f = 5\%$, $k^{E,u} = 15\%$, $n = 4$, $\underline{l} = 0.7$, $D_0 = 500$, $\alpha = 50\%$ and $\tau = 34\%$ and write down both values. Is it fair to evaluate a company financed by book values with WACC?

Hint: Use the formula obtained from Prob. 2.17 in Sect. 2.5.5 for the firm financed by market value. Notice that l_0 is not given.

2.7 Other Financing Policies

We see our task as examining every conceivable finance policy and deriving appropriate valuation equations. In this section we introduce three finance policies, which in our opinion are not as equally significant as those we have dealt with up to now. Yet there will be situations in which the application of one of these three forms is called for. Throughout this section we will assume that debt is risk-free.

2.7.1 Financing Based on Cash Flows

We want to discuss a fourth form of financing policy in this section that is based on the firm's free cash flows. If the free cash flows should happen to be high, then a lot of debt will be paid back. If in contrast, the cash flows turn out to be lower, debt redemption is abstained from. One such form of financing policy seems to us to be fully plausible with high leverage (at least for the time being).

One-Period Financing Policy As far as we know, a financing policy of this kind was only examined twice in the literature until now. That shows that considerable difficulties come up with the establishment of values of firms if it is assumed that financing based on cash flows is carried out over a longer period of time: the value of the firm is then dependent upon the price of certain exotic options. To keep these difficulties at a minimum, we consider a special case. The leverage should be based on free cash flows only in the first year; afterwards the amount of debt may remain constant. This leads to the following definition.[53]

[53] The symbol X^+ is defined as

Definition 2.14 *A firm is financed based on cash flows if the debt develops*

$$\widetilde{D}_t := \left(D_0 - \alpha\left(\widetilde{CF}^l_1 - r_f D_0\right)\right)^+$$

for $t \geq 1$. α is thereby a real number between zero and one, $\alpha \in (0, 1]$.

The definition reads as follows: The future amount of debt is established in that the initial amount is decreased by a random debt service. This random debt repayment is established as a part of that amount, which remains from the first year's free cash flow when the interest due has been subtracted. If the random redemption should be larger than the initial debt, then it is at most as large as this. The maximum condition is required so that negative amounts of debt are avoided.[54]

We can give a valuation equation for this case. To do so we use a put option on the value of the unlevered firm at time $t = 1$ with an exercise price of $\frac{1+\alpha r_f(1-\tau)}{\alpha\frac{1+g}{k^{E,u}-g}} D_0$. This put has the value Π. The following relation is then valid.

Theorem 2.21 *The firm lives until T and follows a financing policy based on cash flows. The cash flows of the unlevered firm are martingale-like. Debt is risk-free. The market value of a levered firm is then established from*

$$V^l_0 = V^u_0 + \frac{\tau r_f D_0}{1+r_f} + \tau\alpha\frac{1+g}{k^{E,u}-g}\left(1 - \frac{1}{\left(1+r_f\right)^{T-1}}\right)\Pi\ .$$

We refer readers, who want to grasp this somewhat difficult formula, to the appendix.[55]

The last theorem clarifies that for the valuation of the levered firm, it is necessary to trade a put with a determined exercise price. If this put is not traded, the valuation will not be successful. It is said in this case that the market is not complete.

Perpetual Annuity In general we cannot most likely assume that the put option required for the valuation of the firm where the financing is based on cash flows is traded. But the last theorem then has no practical relevance for valuation of firms. The derivation of a

$$X^+ = \begin{cases} X & \text{if } X \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

[54] If full distribution is not insisted upon one could interpret negative debt as retained earnings. In this case the maximum condition can be left out which simplifies the calculation. But notice that such a behavior is suboptimal since the company will pay corporate tax on interest obtained.

[55] See Sect. 5.4.

valuation equation that does not have to fall back on options is possible under a broader assumption.

Theorem 2.22 *The conditions of Thm. 2.21 are valid. In addition, the first period's debt is larger than zero, and finally, the expectation of the cash flows are constant. The market value of the levered firm is then established from*

$$V_0^l = \left(1 - \alpha\tau \frac{k^{E,u}}{1+k^{E,u}} \frac{1 - \frac{1}{(1+r_f)^{T-1}}}{1 - \frac{1}{(1+k^{E,u})^{T-1}}}\right) V_0^u + \frac{1 + (1+\alpha(1-\tau))r_f - \frac{1+\alpha(1-\tau)r_f}{(1+r_f)^{T-1}}}{1+r_f} \tau D_0 \ .$$

The proof is again found in the appendix.[56]

2.7.2 Example

We also want to calculate the firm value with financing based on cash flows in our finite example. To do so we suppose that at time $t = 0$ debt is

$$D_0 = 100$$

and the factor α amounts to exactly

$$\alpha = 1 \ .$$

We are concentrating our attention on the put and next establish its exercise price. For this we need the dividend-price relation of the unlevered firm. Since we have already determined the cash flows as well as the values of the firm at time $t = 1$, this is easy to do. We get

$$\frac{1+g}{k^{E,u} - g} = \frac{\widetilde{CF}_1^u(u)}{\widetilde{V}_1^u(u)} = \frac{\widetilde{CF}_1^u(d)}{\widetilde{V}_1^u(d)} \approx 0.5692 \ .$$

The exercise price of the put therefore comes to

$$\frac{1 + \alpha r_f(1-\tau)}{\alpha \frac{1+g}{k^{E,u}-g}} D_0 \approx \frac{1 + 0.1 \times (1-0.5)}{0.5692} 100 \approx 184.48 \ .$$

[56] See Sect. 5.4.

The conditional payments of the put thus amount to

$$\widetilde{\Pi}_1 = \begin{cases} \left(\frac{1+\alpha r_f(1-\tau)}{\alpha \frac{1+g}{k^{E,u}-g}} D_0 - \widetilde{V}_1^u(u) \right)^+ = 0.00 & \text{if up in } t = 1 \\ \left(\frac{1+\alpha r_f(1-\tau)}{\alpha \frac{1+g}{k^{E,u}-g}} D_0 - \widetilde{V}_1^u(d) \right)^+ \approx 26.35 & \text{if down in } t = 1 \,. \end{cases}$$

Since it is supposed that there is no free lunch in the market the Fundamental Theorem of Asset Pricing must hold for the put. This means

$$\Pi = \frac{\mathrm{E}^Q\left[\widetilde{\Pi}_1\right]}{1+r_f}.$$

Employing the appropriate risk-neutral probabilities from Fig. 2.3 gives

$$\Pi \approx \frac{0.0833 \times 0.00 + 0.9167 \times 26.35}{1.1} \approx 21.96 \,.$$

We end up calculating the value of the levered firm with

$$\begin{aligned} V_0^l &= \widetilde{V}_0^u + \frac{\tau r_f D_0}{1+r_f} + \tau\alpha \frac{1+g}{k^{E,u}-g}\left(1 - \frac{1}{\left(1+r_f\right)^{T-1}}\right)\Pi \\ &\approx 229.75 + \frac{0.5 \times 0.1 \times 100}{1.1} + 0.5 \times 0.5692 \times 21.96 \times 0.1736 \approx 235.38 \,. \end{aligned}$$

For the case of an infinitely lived firm financing based on cash flows requires the knowledge of a multiplicity of complicated derivatives. We believe that this assumption is far from being realistic to pursue this case.

2.7.3 Financing Based on Dividends

Distribution and Debt Redemption It is apparent with many corporations, that the managers hold the dividends constant to be paid to the shareholders over a longer period of time. Such a policy has consequences for the firm's amount of debt. It emerges from Fig. 1.1 that there are only two uses for the free cash flow: distribution to the owners, or to serve the creditors with interest and debt repayments. In all of the financing policy variations discussed up to now, the debt redemption was exogenously set and followed a more or less realistic plan. Now we want to look at a new possibility and assume that the management determines the distribution. A look at Fig. 1.1 makes it clear that with a pre-given free cash flow, such a policy has consequences for debt redemption. If the managers subordinate the redemption of debt under the implemented policy of dividends in the way described, then we want to speak of financing based on dividends.

In relation to the equity approach, we made it clear that the shareholders of a levered firm receive payments at time t of (in the case of no insolvency)[57]

[57] See Eq. (2.28).

$$\widetilde{Div}_t = \widetilde{CF}^l_t - \widetilde{D}_{t-1} - \widetilde{I}_t + \widetilde{D}_t.$$

If the firm now tries to manage the amount of debt so that this exactly corresponds to the already determined dividends Div, then the following definition is useful.

Definition 2.15 *A firm is financed based on dividends over n periods, when the development of debt meets the condition*

$$\widetilde{D}_t := \left(Div - \widetilde{CF}^l_t + \widetilde{D}_{t-1} + \widetilde{I}_t\right)^+$$

for all times $t \leq n$. The payments Div are then deterministic and correspond to the distributions to the shareholders at times $t \leq n$.

We could, in an analogous way to the financing based on cash flows, now again prove a theorem, which brings the value of the levered firm in proportion to the value of the unlevered firm and one option. But since we already stressed in the previous section that we regard such valuation equations as useless in practice, we want to look right away here at a special case.

Time Limitation of the Dividends Policy It does not make sense to assume that a policy of constant dividends can be carried out for a very long time. Firstly, that does not agree with the picture that can be empirically observed, and secondly, you fall into a logical contradiction if you assume that constant dividends can be paid for all eternity from a taxable, levered firm. If, namely, the same dividends Div would really be continuously paid, then the market value of the levered firm would come to $\frac{Div}{r_f} + D_0$ at time $t = 0$. The tax advantages bound up with the financing policy would, it is true, be achieved, but never distributed to the shareholders and that could be a contradiction of the condition of transversality.

Constant Rate of Growth The same now applies for the method of financing based on dividends just as it did for the financing policy based on cash flows, namely that without simplifying assumptions we can only derive valuation equations with unpleasant options-terms. For this reason, and only for this reason, we suppose that the amount of debt remains constant beyond time $t = n$, and the expected cash flows of the unlevered firm grow constantly with the rate g.

Theorem 2.23 *The firm implements a financing policy based on dividends over $n \leq T$ periods, the cash flows of the unlevered firm are martingale-like and debt is risk-free. In addition, the amount of capital is continuously larger than zero up to the n^{th} period. The expectation of the cash flows of the unlevered firm grows with the constant rate g. The market value of a levered firm is then established from*

$$V_0^l = \left(1 - \gamma^n \left(1 - \tau \left(1 - \frac{1}{(1+r_f)^{T-n}}\right)\right)\right) D_0$$
$$+ \left(1 - \gamma^n \left(1 - \tau \left(1 - \frac{1}{(1+r_f)^{T-n}}\right)\right) - \tau \left(1 - \frac{1}{(1+r_f)^T}\right)\right) \frac{Div}{r_f(1-\tau)}$$
$$+ \left(\delta^n - \delta^T + \frac{\gamma^n - \delta^n}{\frac{\gamma}{\delta} - 1} \frac{k^{E,u} - g}{1+g} \left(1 - \tau \left(1 - \frac{1}{(1+r_f)^{T-n}}\right)\right)\right) \frac{V_0^u}{1-\delta^T} ,$$

where $\gamma = \frac{1+r_f(1-\tau)}{1+r_f}$ *and* $\delta = \frac{1+g}{1+k^{E,u}}$.

You find the proof in the appendix.[58]

For the case of an infinite lifespan, the above equation is simplified to the extent that the factor δ^T then moves towards zero.

2.7.4 Example

We again fall back upon the payment values of our finite example to establish the value of the firm for the case of financing based on dividends as well. We suppose that the firm at time $t = 1$ distributes a dividend of

$$Div = 150$$

and the firm does not change the amount of debt necessary to do so from $n = 1$ on. According to Def. 2.15, at future time $t = 1$ an amount of debt is required of

$$\widetilde{D}_1 = \left(Div - \widetilde{CF}_1^u + (1 + r_f(1-\tau))\right)^+ = \begin{cases} 145 & \text{if up in } t = 1, \\ 165 & \text{if down in } t = 1. \end{cases}$$

The conditions of the theorem are obviously met with that; debt remains positive. Let us first establish the parameters γ and δ

$$\gamma = \frac{1 + r_f(1-\tau)}{1+r_f} = \frac{1 + 0.1(1-0.5)}{1+0.1} \approx 0.9545 ,$$
$$\delta = \frac{1+g}{1+k^{E,u}} = \frac{1+0.1}{1+0.2} \approx 0.9167 .$$

The value of the levered firm then results from the equation

[58] See Sect. 5.5.

$$V_0^l = \left(1 - \gamma\left(1 - \tau\left(1 - \frac{1}{(1+r_f)^2}\right)\right)\right) D_0 + \left(1 - \gamma\left(1 - \tau\left(1 - \frac{1}{(1+r_f)^2}\right)\right) - \tau\left(1 - \frac{1}{(1+r_f)^3}\right)\right) \frac{Div}{r_f(1-\tau)} + \left(\delta - \delta^3 + \frac{\gamma - \delta}{\frac{\gamma}{\delta} - 1} \frac{k^{E,u} - g}{1+g} \left(1 - \tau\left(1 - \frac{1}{(1+r_f)^2}\right)\right)\right) \frac{V_0^u}{1 - \delta^3} .$$

Entering all values known to us results in

$$V_0^l \approx 237.50 .$$

We already mentioned that a perpetual constant dividend could contradict transversality. That is why we will not evaluate our infinite example here. See the problem set for another infinite example.

2.7.5 Financing Based on Debt–Cash Flow Ratio

Dynamic Leverage Ratio The dynamic leverage ratio is a real number by which the cash flow is set in relation to the firm's debts,

$$L_t^d = \frac{\widetilde{D}_t}{\widetilde{CF}_t^l}. \tag{2.36}$$

This ratio serves as a (simpler) criterion for the length of time in which the firm would be completely self-financed only using cash flows for debt redemption. We want to use this ratio to look at a sixth financing policy.

Definition 2.16 *A firm is financed based on debt-cash flow ratios if these ratios are deterministic.*

If a firm follows this debt schedule the following theorem can be verified. The proof can be found in the appendix.[59]

Theorem 2.24 (Debt-cash flow ratio) *The firm implements a financing policy based on debt-cash flow ratios. The cash flows of the unlevered firm are martingale-*

[59] See Sect. 5.6.

like and debt is risk-free. The market value of a levered firm is then established from

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \widetilde{D}_t \sum_{s=t}^{T-1} \widetilde{L}_s^d \ldots \widetilde{L}_{t+1}^d \left(\frac{\tau r_f}{1+r_f}\right)^{s+1-t} + \sum_{s=t+1}^{T-1} \left(\sum_{u=s}^{T-1} \widetilde{L}_u^d \ldots \widetilde{L}_s^d \left(\frac{\tau r_f}{1+r_f}\right)^{u+1-s}\right) \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{(1+k^{E,u})^{s-t}}$$

where for the product $\widetilde{L}_s^d \ldots \widetilde{L}_{t+1}^d = 1$ *holds if* $s = t$.

Infinite Lifetime If the firm exists infinitely long and if the dynamic leverage ratio remains constant the following theorem can be shown.

Theorem 2.25 (Debt-cash flow ratio in infinite lifetime) *The assumptions of Thm. 2.24 are valid. If the firm has an infinite lifetime and if the debt-cash flow ratio remains constant, then the firm value is given by*

$$\widetilde{V}_t^l = \left(1 + \frac{\tau r_f \widetilde{L}^d}{1 + r_f(1-\tau\widetilde{L}^d)}\right)\widetilde{V}_t^u + \frac{\tau r_f}{1 + r_f(1-\tau\widetilde{L}^d)}\widetilde{D}_t\ .$$

2.7.6 Examples

2.7.6.1 The Finite Case (Continued)

The debt-cash flow ratio will be constant

$$\widetilde{L}^d = 1\ .$$

Debt at time $t = 0$ is

$$D_0 = 100\ .$$

From the above theorems we have

$$V_0^l = V_0^u + D_0 \left(\frac{\tau r_f}{1+r_f} + \left(\frac{\tau r_f}{1+r_f}\right)^2 + \left(\frac{\tau r_f}{1+r_f}\right)^3 \right) + \left(\frac{\tau r_f}{1+r_f} + \left(\frac{\tau r_f}{1+r_f}\right)^2 \right) \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}} + \frac{\tau r_f}{1+r_f} \frac{\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1+k^{E,u})^2} .$$

This gives

$$\begin{aligned} V_0^l &= 229.75 + 100 \left(\frac{0.5\times 0.1}{1+0.1} + \left(\frac{0.5\times 0.1}{1+0.1}\right)^2 + \left(\frac{0.5\times 0.1}{1+0.1}\right)^3 \right) \\ &\quad + \left(\frac{0.5\times 0.1}{1+0.1} + \left(\frac{0.5\times 0.1}{1+0.1}\right)^2 \right) \frac{100}{1+0.2} + \frac{0.5\times 0.1}{1+0.1} \frac{110}{(1+0.2)^2} \\ &\approx 241.94 \end{aligned}$$

for the value of the levered firm.

2.7.6.2 The Infinite Case (Continued)

If the firm exists infinitely long and maintains a debt-cash flow ratio of

$$\widetilde{L}^d = 1$$

then with debt of 100 at $t = 0$ the levered firm is worth

$$\begin{aligned} V_0^l &= \left(1 + \frac{\tau r_f \widetilde{L}^d}{1 + r_f(1-\tau \widetilde{L}^d)}\right) \widetilde{V}_t^u + \frac{\tau r_f}{1 + r_f(1-\tau\widetilde{L}^d)} \widetilde{D}_t \\ &= \left(1 + \frac{0.5\times 0.1\times 1}{1+0.1(1-0.5\times 1)}\right) 500 + \frac{0.5\times 0.1}{1+0.1(1-0.5\times 1)} 100 \\ &\approx 528.57 . \end{aligned}$$

2.7.7 Comparing Alternative Forms of Financing

In the previous sections, we discussed different forms of financing and their influences on the value of firms. In the case of autonomous financing, the analyst knows the firm's future amount of debt $D_0, D_1, \ldots$ A valuation equation, which is possible under this assumption and delivers the correct value of the firm is the APV equation. If in contrast financing is based on market values, the analyst knows the firm's future debt ratios $l_0, l_1, \ldots$ measured in market values. A valuation equation that results in the correct value of the firm under this condition is the WACC formula. In the case of financing based on book values, the future debt ratios $\underline{l}_0, \underline{l}_1, \ldots$ measured in book values are known to the analyst. Which valuation equation is applied under this condition is dependent upon whether the firm follows a policy of full distribution, only takes on replacement investments or conditions its investments upon attained cash flows. With financing based on cash flows, the firm reduces its amount of debt (for a limited time) by a fixed proportion of its free cash flows. Special valuation equations can be given that bear just this sort of financing calculation. Dividend-based financing is characterized by a firm paying constant dividends over an extended period of time.

Extent of Differences in Value In order to illustrate the way the different valuation equations work, we have used a standard example. Every reader, who has paid attention to our calculations can determine that the respective values of firms are not that dramatically different from each other. That brings up the question as to the practical relevance of those valuation equations either given or developed by us. It is after all conceivable that the differences in value of the forms of finance specified by us are relatively small. In this case, using any valuation formula you like and simply accepting the possible resultant valuation mistakes can be economically justified. It is, however, not now clear to us at all how you would have to go about confirming or refuting such an assertion if the firm lives longer than three periods or possesses different cash flows.

APV and WACC Autonomous financing and financing based on market values are particularly prominent forms of financing in the DCF literature, which is why we still want to spend some time on them here. In the first case we recommend the APV formula, and in the second case the WACC formula. "Mixed formulas" are derived in the literature that look like WACC formulas and are nevertheless appropriate to be applied for autonomous financing, or that look like APV formulas and still can be used for value-based financing.[60] From an academic viewpoint, such valuation formulas may be interesting, but they are not practically relevant. Under an autonomous approach, the analyst assumes future debt amounts to be known with certainty. What sense is there then of the fiction of not knowing the amounts of debt (otherwise only the APV formula would be needed in order to value), but instead falling back upon the expected debt ratios in order to enter them into a WACC formula? It is likewise so for the opposite case. If an investor assumes deterministic future debt ratios, why should she pretend not to know them and instead use the expected amount of debt in an APV formula? Whoever wants to get from A to B can either take the direct path or take the long way round. Economists normally avoid such long ways. We find it even more strange at the least to propagate such round about ways.

With autonomous financing, the tax advantages implied by the credit terms are deterministic, whereas under financing based on market values they are stochastic. Even if the expectations are identical to the tax savings[61], from the perspective of risk-averse investors, risk-free payments are always worth more than risky payments. It follows that the two assumptions need not lead to identical firm values—a fact that is often

[60] See Wallmeier (1999) or Hillier et al. (2008, Chapt. 13.2).

[61] This is not the case in our example. This explains why the firm's value under the WACC approach exceeds its APV value, even though the tax advantages under WACC are subject to uncertainty.

underemphasized in the literature. In practice, when applied consistently, WACC and APV produce distinctly different firm valuations.

2.7.8 Problems

Problem 2.20 It is not necessary the case that the assumptions of Theorem 2.23 must violate transversality: show that for $n = T - 1$ and $n \to \infty$ the value of the levered firm satisfies

$$\lim_{n\to\infty} V_0^l = D_0 + \frac{Div}{r_f}$$

which is very intuitive since the company will pay a given dividend to the shareholders at any time in the future.

Problem 2.21 Assume an unlevered company has a lifetime $T = 2$ and its cash flows are given by the Fig. 2.13. Ignore insolvency. The unlevered cost of capital is $k^{E,u} = 20\%$, any up or down movement has a probability 0.5. The risk-free interest rate is $r_f = 10\%$, the tax rate is $\tau = 34\%$. The levered company has debt $D_0 = 0$ at time $t = 0$.

a) Write down a formula for the value of the levered firm if $\widetilde{D}_1(u)$ and $\widetilde{D}_1(d)$ are given.
b) Assume the firm has an expected debt $\mathrm{E}[\widetilde{D}_1] = 100$. Which debt schedule (i.e., what values of $\widetilde{D}_1(u)$ and $\widetilde{D}_1(d)$) yields the highest value of the levered firm?

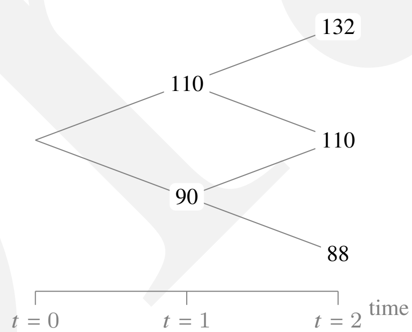


Fig. 2.13 Cash flows in Prob. 2.21.

2.8 Further Reading

Research on the valuation of tax advantages dates back more than seven decades; see Modigliani and Miller (1958) and Modigliani and Miller (1963). In these studies, the authors also examined the case of a perpetual annuity. Sick (1990) was the first to make extensive use of certainty equivalents (i.e., a martingale technique) to derive valuation formulas incorporating taxes, although Ross (1987) had addressed the issue primarily within a one-period framework.

The well-known formula for a perpetual rent first appeared in Williams (1938, p. 72), yet it is still referred to as the Gordon–Shapiro formula. Neither Williams nor Gordon and Shapiro considered uncertainty.

The first two financing policies discussed in this chapter can be found in nearly every textbook on corporate finance; see, for instance, Brealey et al. (2025, Sect. 18-4 (APV) and Sect. 18.1 (WACC)). We believe that the term "autonomous" originates from Richter (1998).

Autonomous financing with a non-constant amount of debt was first presented in Myers (1974). Financing based on market values was examined by Miles and Ezzell (1980) and Miles and Ezzell (1985). However, the proof that now bears their names could only be established under the assumption of a constant leverage rate. A generalization was later provided by Löffler (2004). Arnold et al. (2018) derived results for a financing policy in which refinancing dates occur at multiples of a number greater than one ("delayed financing"). Dierkes and Schäfer (2016) analyzed a policy in which a firm combines both capital structure targets and predetermined debt levels. Finally, Barbi (2012) employed risk-neutral probabilities to derive general results for the tax shield.

Although several publications on market-value-based financing appeared later—Harris and Pringle (1985) and Clubb and Doran (1995) may serve as representative examples—they did not introduce any new financing concepts. Financing based on cash flows was examined by Arzac (1996) and Löffler (2000).

Martingales are covered in any textbook on time series analysis. Hamilton (1994) or Brockwell and Davis (2016) are a good source. Martingale-like cash flows were introduced in a more general form by Lintner (1956) and Rubinstein (1976). Fama (1977) was likely the first to discuss martingale-like cash flows and discount rates systematically in the context of valuation. In Barberis et al. (1998) cash flows that are not martingale-like were considered, although in a very different context. The implications of the martingale property for the cost of capital and discount rates were first examined systematically by Laitenberger and Löffler (2006). Löffler (2002) points out the problem of the Modigliani-Miller adjustment. In an earlier discussion Fernández (2004), Fieten et al. (2005) and Cooper and Nyborg (2006) clarify the relation between Modigliani-Miller and Miles-Ezzell. The observations on financing based on book values are new, even if individual elements, such as the operating assets relation, are already known in the literature. Compare for instance Feltham and Ohlson (1995, S. 693f.) or Penman (2012, chapter 8).

The literature cited so far concentrated on discrete-time models. There is a vast literature on continuous-time models for the firm which we will not mention here.

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Chapter 3
Personal Income Tax

Abstract This chapter develops a valuation framework for settings with personal income taxation.
We again distinguish unlevered and levered firms, but not to capture debt effects; rather, the focus is on earnings retention as the source of tax advantages. We discuss and evaluate several retention policies, and examine whether DCF methods can shed light on how the cost of capital responds to changes in underlying tax rates.
Finally, we show that a simple relation between the cost of capital and tax rates—often used in practice—can create arbitrage opportunities.

We now shift gears. In the previous chapter we assumed the firm was taxed and the financiers were tax-exempt. We now assume the reverse: the financiers are liable to income tax, while the firm is untaxed.

Certainly not all readers will think it sensible, in valuing firms, to take taxes due at the financiers' level into account. Textbooks, at any rate, tend to omit income tax altogether.[1] If, however, one keeps in mind that income tax affects private investors' consumption in all cases, there is little reason not to consider it in valuation. Someone who acquires a firm will report different figures on her income tax return than someone who invests in the capital market. This alone argues for including income tax in firm valuation. The German profession of certified public accountants, for example, decided in 1997 to endorse the consideration of income tax in valuation.[2]

The reader can expect a roadmap in this chapter that closely mirrors the plan of action in the previous chapter. We will soon see that, despite clear differences, corporate income tax and personal income tax share enough structure to justify treating them in parallel. Accordingly—and perhaps to some readers' surprise—we will again speak of levered and unlevered firms. We use the same symbol for the tax rate and revisit the various financing policies to value the tax advantages of a levered firm relative to an unlevered firm.

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[1] Income tax is typically not treated in detail. In the first edition of our book we could at least point to some textbooks that covered the perpetual-annuity case. In preparing the second edition, we found that even these cases had disappeared from the literature. It seems that the inclusion of personal taxes has fallen out of favor.

[2] See Institut der Wirtschaftsprüfer in Deutschland (2000).

We will hardly surprise many readers by noting that including personal income tax in the theory of firm valuation breaks considerable new ground. The first work on WACC and APV appeared more than half a century ago. By contrast, the literature on firm valuation has often ignored taxes at the investors' level up to the present, leaving much less prior work to build on. Accordingly, this chapter cannot offer a systematic survey of established results; instead, we develop new ones. Given the scarcity of comparable studies, we cannot readily benchmark our findings against existing papers. We therefore present this chapter not as a synthesis of settled knowledge but as a contribution to the theoretical discussion of personal income taxation within DCF-based firm valuation.

3.1 Unlevered and Levered Firms

DCF theory, in essence, repeatedly addresses how tax shields should be valued. For discussion of a tax advantage (or disadvantage) to be economically meaningful, a reference point is needed against which the advantage can be measured. That reference point is a firm that follows a specific policy—an unlevered firm.

3.1.1 'Leverage' Interpreted Anew

Do you recall the beginning of the previous chapter? There we assumed that the firm pays taxes while the financiers are exempt. We showed that a levered firm bears a lighter tax burden than an unlevered one. We also noted that one can understand what an unlevered firm is without further detail, whereas a precise description of a levered firm requires substantially more information.[3] At the outset of that chapter, we agreed to call the firm without debt unlevered and to describe the indebted firm as levered.

Reference Firm We will proceed analogously in this chapter. We now face a different tax setting. The firm remains exempt from tax, while the financiers are liable to income tax. For income-tax purposes, dividends and interest are taxable; hence a firm with a policy of full distribution gives rise to a large personal tax burden. By contrast, if the firm distributes only part of its profits to financiers, an income-tax saving is to be expected. Retaining profits creates a tax shield for investors. It is therefore natural to use the full-distribution firm as the reference case, since no other distribution policy makes the discussion equally clear. By contrast, anyone valuing a firm with partial distribution must specify precisely which share of the cash flows is retained in which periods. Full distribution is unambiguous. Because full distribution entails more personal income tax than reduced distribution, we shall call the former an unlevered firm and the latter a levered firm.

Please note that we use the term levered differently in this chapter than in the preceding one. Here, levered no longer means "indebted," but rather "partial distribution." The

[3] See Sect. 2.1.

firm's capital structure plays no role in this chapter. In this chapter, leverage relates solely to whether, and how much, is retained in the firm. Although this may be somewhat irritating for readers, we have deliberately chosen this terminology to make use of structural similarities with the previous chapter.

In the previous chapter, a full retention policy was assigned to both the unlevered and the levered firm, while all other possible retention policies lay outside the scope of our discussion. In this chapter, we proceed in a similar manner. Both firms—the unlevered (or fully distributing) firm and the levered (or partially distributing) firm—are assumed to be self-financed and therefore without debt. Financing policy, accordingly, is not the subject of this discussion. The question of how financing and retention policy can be linked will be taken up in the final chapter. Seen in this light, the unlevered firm of the previous chapter and the one considered here are identical: both are debt-free and both fully distribute their cash flow to their owners.

If we interpret the notation anew in the sense explained here, then it should also apply that a firm with full distribution has the same value as a firm with partial distribution, as long as no taxes are imposed. That agrees with the theorem of Miller and Modigliani (1961) on the irrelevance of dividend policy. This theorem says that it does not matter when a firm distributes its earnings so long as taxes do not play a role.

Notation We denote the market value of the unlevered firm by $\widetilde{V}_t^u$ and that of the levered firm by $\widetilde{V}_t^l$. Note that neither firm carries debt; hence both values equal the market value of equity. Analogously, we use $\widetilde{CF}_t^u$ for the post-tax free cash flows of the full-distribution firm and $\widetilde{CF}_t^l$ for those of the partial-distribution firm. Correspondingly, $\widetilde{Tax}_t^u$ denotes the shareholders' taxes under full distribution, whereas $\widetilde{Tax}_t^l$ denotes the shareholders' income taxes under partial distribution.

Note another important difference between the previous and present chapters. Previously, $\widetilde{CF}_t^l$ denoted the free cash flow accruing to all investors (shareholders and debtholders). Now there are no debtholders. In contrast to the previous chapter, the free cash flow here consists solely of payments to shareholders.

Positive Dividends If we proceed on the assumption that a firm's cash flows are the pool from which dividends to owners are paid, an additional restriction arises. In the previous chapter, when cash flows turned negative, this simply meant that financiers injected additional equity. If that was no longer possible, we called the situation insolvency and discussed its implications in detail. Negative payments, however, make no sense for dividends. We therefore assume, in what follows, that the cash flows of both the levered and the unlevered firm are large enough that dividends never become negative.

3.1.2 The Unlevered Firm

In this chapter, firms with full distribution play the same role as debt-free firms did in the chapter on corporate taxes. We assume that such firms are just as rare in practice as entirely self-financed firms. Nevertheless, it is important to be able to value them. Just as we argued previously that an indebted firm can be valued only if one can also value a self-financed firm, we can now maintain that a firm with partial distribution can be valued only if there is a method for valuing a firm with full distribution.

If the cost of equity and the free cash flows of a firm with full distribution are known, the valuation equation follows directly. We begin by defining the cost of equity.

Definition 3.1 (Cost of equity) *Cost of equity $\widetilde{k}_t^{E,u}$ of an unlevered firm is the conditional expected return*

$$\widetilde{k}_t^{E,u} := \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u + \widetilde{V}_{t+1}^u\right]}{\widetilde{V}_t^u} - 1 .$$

Since this cost of equity is mathematically identical to the cost of equity of an unlevered firm (Def. 2.1), the valuation equation for a firm with full distribution follows immediately. If we again assume that the cost of equity is deterministic and time-invariant, then the valuation equation is exactly the same as the corresponding one for a fully self-financed firm in Thm. 2.1,

Theorem 3.1 (Market value of the unlevered firm) *If the cost of equity of the unlevered firm $k_t^{E,u}$ is deterministic, then the value of the firm, which fully distributes its free cash flows, amounts at time t to*

$$\widetilde{V}_t^u = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{\left(1 + k^{E,u}\right)^{s-t}} .$$

We do, however, consider it important to emphasize a point that may take some getting used to. The cost of equity $k^{E,u}$ used here is *post-tax*, not *pre-tax*. Equally important, we refrain from making any claims about the link between pre-tax and post-tax cost of equity; instead, we take the post-tax cost of equity as given. We will return to the likely relation between the two in more detail later.[4]

In the chapter on corporate income tax, we relied on a premise we termed the assumption of martingale-like cash flows.[5] To develop valuation equations—and, above all, adjustment equations—we repeatedly used Thm. 2.3. We will need a corresponding theorem when dealing with personal income tax. To obtain it, we first restate the martingale-like cash flow assumption. Thus

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[4] See Sect. 3.2.

[5] See Assump. 2.1.

Assumption 3.1 (Martingale-like cash flows) *There is a real number g such that*

$$\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] = (1+g)\,\widetilde{CF}^u_t$$

is valid for the unlevered firm's cash flows.

Since we have already discussed the economic significance of this assumption, we need not repeat it. Before we can prove specific valuation equations, we must complete a few preparatory steps. First, we describe in more detail the tax at the center of this chapter. We must also address what happens to the Fundamental Theorem of Asset Pricing when personal income tax is taken into account.

3.1.3 Income and Taxes

Economists usually describe a type of tax by specifying who bears the liability, how the tax base is defined, and which tax rate schedule applies. In our setting, the taxpayers are individuals.

Categories of Income In most countries, income tax is assessed on a base that is typically difficult to compute, because detailed statutory provisions must be followed. The core of this base is the sum of recognized income categories. For the moment, we distinguish between the income of owners and that of creditors, although no creditors appear in our model yet.

1. Owners' income can refer either to the firm's earnings or to the firm's dividends. If the firm's earnings are taxed, regardless of whether they are distributed or retained, then "accrued income" forms the tax base. If, by contrast, the cash that shareholders receive is taxed, this is "realized income". Income from shares traded on capital markets is always included in the second group.
2. When we speak of creditors' income, we mean interest. In many countries, interest income and dividend income are taxed differently.

Redemption of Capital Owners sometimes receive payments that are not dividends—for example, share capital reductions or payments on liquidation. These should not be confused with dividends, as they are generally not subject to income tax.

Earnings Retention It is possible to retain part of the distributable cash flow within the firm. Let $\widetilde{A}_t$ denote these retention amounts. The retention amounts are always non-negative. We assume they are invested by the firm for one period at the (to be specified) interest rate $\widetilde{r}_t$ and then distributed to the financiers.

Table 3.1 shows how to derive levered taxable income from the firm's pre-tax gross cash flows. This yields the expression in the last line of the table for the financiers' taxable income. The owners must, however, still pay income taxes on this amount. Accordingly, both the unlevered and the levered free cash flows equal the respective taxable income minus the tax payments.

Table 3.1 From pre-tax gross cash flow to income.

	Pre-tax gross cash flow	$\widetilde{GCF}_t$
−	Investment expenses	$\widetilde{Inv}_t$
=	Shareholder's unlevered taxable income	$\widetilde{GCF}_t - \widetilde{Inv}_t$
−	Retained earnings	$\widetilde{A}_t$
+	Cash flow from retained earnings	$(1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}$
=	Shareholder's levered taxable income	$\widetilde{GCF}_t - \widetilde{Inv}_t$ $-\widetilde{A}_t + (1+\widetilde{r}_{t-1})\widetilde{A}_{t-1}$

In this figure, the gross cash flows and investment outlays are identical for the levered and the unlevered firm. The unlevered firm, however, has no retained earnings.

Yield of the Retained Earnings Amounts Under what conditions does the firm invest the amount $\widetilde{A}_t$? Recall that managers—regardless of distribution policy—should undertake every investment project with a positive net present value. In our framework, using $\widetilde{A}_t$ for operating investment is ruled out, as is using it for the repayment of principal or for writing down equity. The only remaining possibility is to invest $\widetilde{A}_t$ on the capital market for one period at the rate $\widetilde{r}_t$.

Investing on the capital market may be risk-free or risky. One might expect that the choice would matter. However, if personal income taxes are not (yet) taken into account, the Fundamental Theorem implies that we need not distinguish between these cases. Let r_f denote the risk-free interest rate before income tax. If the amount $\widetilde{A}_t$ is invested in risk-free assets, then the following holds

$$\widetilde{r}_t = r_f \ . \tag{3.1}$$

In the case of a risky investment, the firm invests the amount $\widetilde{A}_t$ and receives $(1+\widetilde{r}_t)\widetilde{A}_t$ one period later. If capital markets are arbitrage-free, the payoff at $t+1$ must be valued at its time-t present value. We therefore invoke the Fundamental Theorem of Asset Pricing. Because the investment is made by the firm and, under our assumptions, the firm pays no taxes, we use Thm. 1.2, which applies in a no-corporate-tax setting. It follows that

$$\widetilde{A}_t = \frac{\mathrm{E}_t^Q\left[(1+\widetilde{r}_t)\widetilde{A}_t\right]}{1+r_f} \ .$$

With Rules 2 and 5, we then get

$$\mathrm{E}_t^Q\left[\widetilde{r}_t\right] = r_f \ . \tag{3.2}$$

That is a generalization of Eq. (3.1). In the following we will proceed from this relation.

Tax Rates for Interest and Dividends Analogous to corporate taxation, the personal tax rate is linear—there are no exemptions or thresholds. As in the preceding section, the tax rate is assumed known and constant. We have already emphasized that this is a strong, but necessary, assumption. Tax rates on dividends and interest may differ; we denote the dividend tax rate by τ^D and the interest tax rate by τ^I. Although our firms are self-financed, this distinction is relevant for the proof of the Fundamental Theorem, as will become clear shortly.

We need to consider the following. Suppose an investor has an amount G available. There are two riskless ways to invest it. First, the investor can invest personally and, over one period, end up with $G + (1 - \tau^I)\, r_f G$ after paying income tax on interest. Alternatively, the investor can contribute G to the firm and have the firm invest it. In that case, the firm's assets grow to $G + r_f G$ over the next period; when this amount is distributed to the owner, the tax treatment of the interest component is decisive. If that component is taxed as a dividend, the investor receives $G + (1 - \tau^D)\, r_f G$ after tax. This generally differs from $G + (1 - \tau^I)\, r_f G$ (unless $\tau^D = \tau^I$), creating an arbitrage opportunity even with self-financed firms and thereby undermining the model. To avoid this, we are left with two options.

1. We could assume that
$$\tau^I = \tau^D$$
applies.[6] Unfortunately, it is easily verified empirically that many industrial countries do not satisfy the identity we assumed, so we will not consider this option.
2. The second possibility is to define the income-tax base differently from our arbitrage example. For this purpose, we define a capital market that reflects these considerations more precisely. We assume a finite set of tradable assets, which we call basic assets.[7] One of these basic assets is risk-free; the others are risky. Additionally, every firm and every investor may hold only portfolios of these basic assets.

The formulation of this tax base is therefore crucial for eliminating arbitrage opportunities. Any capital-market investment is spanned by portfolios of the basic assets. Moreover, cash flows from the risk-free asset are always taxed at rate τ^I, whereas cash flows from risky assets are taxed at rate τ^D. This definition has far-reaching consequences. Suppose an investor chooses to invest risk-free through a firm rather than privately. When the returns on this investment are distributed, they are taxed as interest rather than as dividends, because the cash flow ultimately derives from the risk-free basic asset. With this definition of the tax base, the arbitrage opportunity described above is eliminated.

Tax Equation Our model's income tax equation for shareholders can now be written principally in the form
$$\widetilde{Tax}_t^u = \tau^D \left(\widetilde{GCF}_t - \widetilde{Inv}_t\right)$$
for the unlevered firm.

[6] This assumption is found in Miller (1977) for instance.

[7] The use of basic assets follows common practice in the mathematical finance literature; see Shreve (2004, Sect.1.2).

As for the levered firm, the matter is less straightforward. The treatment depends on how the amount $\widetilde{A}_t$ is invested: a risk-free placement is taxed differently than an investment in risky securities. In this chapter, we assume the retention is invested in risky assets. The owners of the levered firm then pay taxes in the amount of[8]

$$\widetilde{Tax}_t^l = \tau^D \left(\widetilde{GCF}_t - \widetilde{Inv}_t - \widetilde{A}_t + (1 + \widetilde{r}_{t-1})\widetilde{A}_{t-1} \right) .$$

The difference in the two firms' tax payments is an amount that we refer to as the tax shield, in line with the discussion in the chapter on corporate taxation.

Tax Shield If we want to determine the cash flows of the levered firm in t, we do not only have to observe the earnings retention at time t, but also the earnings retention from the previous period. In total the tax shield amounts to

$$\widetilde{Tax}_t^l - \widetilde{Tax}_t^u = \tau^D \left(-\widetilde{A}_t + (1 + \widetilde{r}_{t-1})\widetilde{A}_{t-1} \right) .$$

To compute the difference in free cash flows between the two firms, note that gross cash flows, principal repayments, and investment outlays are identical in both. The only difference is that one firm retains no earnings, while the other does, which leads to different amounts of income tax. Hence, we can focus entirely on retained earnings and tax payments when calculating the difference between the levered and unlevered free cash flows,

$$\begin{aligned} \widetilde{CF}_t^l - \widetilde{CF}_t^u &= \left(\ldots - \widetilde{A}_t + (1 + \widetilde{r}_{t-1})\widetilde{A}_{t-1} - \widetilde{Tax}_t^l \right) - \left(\ldots - \widetilde{Tax}_t^u \right) \\ &= \left(1 - \tau^D\right) \left((1 + \widetilde{r}_{t-1})\widetilde{A}_{t-1} - \widetilde{A}_t \right) . \end{aligned}$$

With the help of (3.2) and Rule 5 we get for the expectation under risk-neutral probability

$$\mathrm{E}_{t-1}^Q \left[\widetilde{CF}_t^l - \widetilde{CF}_t^u \right] = \left(1 - \tau^D\right) \left(1 + r_f\right) \widetilde{A}_{t-1} - \left(1 - \tau^D\right) \mathrm{E}_{t-1}^Q \left[\widetilde{A}_t \right] . \qquad (3.3)$$

A tax shield arises when the firm does not fully distribute its free cash flows.

[8] If the retention is invested risk-free, the equation changes slightly. "Retention for a risk-free investment" simply means the portfolio contains more risk-free securities. Consequently, the income taxed at rate τ^I decreases accordingly; in the next period it increases again through repayment of the retention and the interest earned. Thus we obtain

$$\widetilde{Tax}_t^l \;=\; \tau^D \left(\widetilde{GCF}_t - \widetilde{Inv}_t \right) \;+\; \tau^I \left(-\widetilde{A}_t + (1 + r_f)\widetilde{A}_{t-1} \right) .$$

3.1.4 Fundamental Theorem

In the last chapter we made thorough use of the Fundamental Theorem of Asset Pricing. We had already introduced this theorem in the first chapter of this book, since it is of such central importance for the derivation of valuation equations.

The Fundamental Theorem says that, under an arbitrage-free capital market, risk-neutral probabilities Q exist. Risk-neutral expectations can thus be discounted in a world without taxes at the risk-free interest rate r_f. This remains true if a corporate income tax is introduced, because when taxation applies only at the firm level, the risk-free rate is identical pre- and post-tax. Here, however, taxation applies at the financiers' level. A financier who invests risk-free and is liable to tax no longer earns a net return of r_f but instead $r_f(1-\tau^I)$. What becomes of the Fundamental Theorem under these conditions? Do risk-neutral probabilities Q still exist? If so, how should risk-neutral expectations be discounted?

For the value of any discretionary portfolio from risky and risk-free assets as well the following theorem now applies.

Theorem 3.2 (Fundamental Theorem with Different Taxation) *If the capital market with a personal income tax is free of arbitrage, the conditional probabilities Q can be chosen to the extent that the following result is valid*

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}^u_{t+1} + \widetilde{V}_{t+1}\right]}{1 + r_f\left(1-\tau^I\right)} .$$

The tax rate on interest income appears in the denominator, even when the portfolio contains risky assets. This result is neither trivial nor immediately evident.[9] We now proceed to develop results analogous to those established in the previous two chapters.

Theorem 3.3 (Williams/Gordon-Shapiro formula) *If the cost of equity is deterministic and the cash flows are martingale-like, then for the value of the unlevered firm*

$$\widetilde{V}^u_t = \frac{1+g}{k^{E,u}-g}\widetilde{CF}^u_t$$

holds.

[9] For a proof, see Becker and Löffler (2024) (in a risk-free setup but with a more general tax function) or the first edition of this book.

Theorem 3.4 (Equivalence of the valuation concepts) *If the cost of equity is deterministic and the cash flows are martingale-like, then the following is valid for all times $s > t$*

$$\frac{\mathrm{E}_t^Q\left[\widetilde{CF}_s^u\right]}{\left(1 + r_f\left(1 - \tau^I\right)\right)^{s-t}} = \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{\left(1 + k^{E,u}\right)^{s-t}} .$$

For the value of equity of the unlevered firm, the cost of equity and the discount rate coincide. However, the taxed interest rate $r_f(1-\tau^I)$ now appears in place of the risk-free rate r_f. The cost of equity $k^{E,u}$ we consider is also a post-tax variable. We nevertheless avoid stating an explicit relation to the levered firm's pre-tax cost of equity here.

We need not prove the two theorems again here. Instead, we refer readers to the relevant pages in the chapter on corporate tax.[10]

3.1.5 What If Tax Scales Are Nonlinear?

Looking at the literature on the effect of income taxation on security prices, one almost invariably finds the assumption of a constant tax rate. This corresponds to a tax liability function that is linear in the tax base. Such an assumption is made for reasons of convenience, as it allows the same formal apparatus to be applied as in the case where taxes are entirely disregarded. We, too, have so far relied on this linearity assumption.

Tax schedules, however, are determined by legislators, not by academic abstraction. As we noted in the opening sections of this book, there is hardly any country in the world where linear tax schedules apply—at least, we are not aware of one. The very existence of tax allowances is already incompatible with the assumption of linearity. Thus, assuming linearity in tax liability functions is fundamentally questionable. If one must instead proceed from nonlinear schedules, the question arises how a firm should then be valued.

In addressing this question, we turn to Becker and Löffler (2024), in which the authors examine how to handle nonlinear income tax schedules. We believe that their results may be of interest for the analysis pursued in this chapter. In Germany, the assumption of constant tax rates is particularly problematic, since a quadratic function of the tax base is applied.[11]

At the outset, however, one crucial point must be emphasized. We have consistently argued that a useful theory of firm valuation must, in principle, be based on stochastic future cash flows. By contrast, the analysis in Becker and Löffler (2024) is conducted entirely under the assumption of deterministic cash flows. Whether their results remain

[10] See Sect. 5.1.

[11] We have discussed this issue in a German paper, the contents of which we shall briefly outline below.

valid once uncertainty is introduced cannot simply be assumed but must be proven—something that has not yet been achieved. For this reason, caution is warranted when transferring their findings. Nevertheless, we do have indications that the results obtained under certainty may provide guidance as to what outcomes are to be expected under uncertainty. We therefore consider it worthwhile to engage here with the results of Becker and Löffler (2024).

It is clear that the firm generates pre-tax cash flows of CF_t.[12] Less clear, however, is how much of this amount is captured by income tax at a given point in time. This is determined by the legislator. The object of taxation is not usually the free cash flow itself, but the so-called tax base, denoted by B_t. Post-tax cash flows are then

$$CF_t^{\text{post-tax}} = CF_t - \text{Tax}(B_t) \ , \tag{3.4}$$

where $\text{Tax}(\cdot)$ represents the nonlinear tax schedule. We therefore assume that, in addition to all cash flows, the relevant tax bases B_t are known at every point in time.

In the context of nonlinear taxation, one usually defines the marginal tax rate. For a given tax base, it specifies how much an additional unit of the base is taxed. For differentiable tax functions, the marginal tax rate is conventionally represented by the first derivative $\text{Tax}'(\cdot)$.[13] In actual tax systems, the marginal tax rate typically lies between 0% and 100%. The marginal tax function will play an important role later on.

Becker and Löffler (2024) assume that the tax liability function $\text{Tax}(\cdot)$ is not linear but convex. Convexity is a topic in mathematics that is not always covered in standard training. It is important to note, however, that convex tax functions are common in many countries. Worldwide, it is standard practice for tax rates to rise with the size of the tax base, a structure usually referred to as a progressive schedule. Although there is a minor distinction between a progressive schedule and a convex tax liability (for details see Becker and Löffler (2024, Appendix 1)), one may reasonably argue that assuming a convex tax liability function is plausible.

For economists, convex tax liabilities can also be illustrated in another way. In many economies, married couples may choose between two methods of taxation:

Separate taxation Each spouse is taxed individually, and the couple's total tax liability is simply the sum of the two.

Joint filing First, the spouses' incomes are added together and then divided by two. The tax liability is calculated on the basis of this halved income, and the resulting amount is doubled.

If both spouses earn identical incomes, both yield the same result. If their incomes differ, however, joint filing produces a lower overall tax burden.

[12] Since Becker and Löffler consider taxation only under certainty, we dispense with the tilde here and in what follows.

[13] A serious technical difficulty with nonlinear taxes is that typical real-world tax functions are often not differentiable, as they lack a well-defined first derivative. Consider, for example, a linear tax function with an allowance: while linear functions are differentiable, at the allowance point no unique first derivative exists—the function has a kink. The paper by Becker and Löffler (2024) discusses these technical issues in more detail; in such cases one must work with the so-called subdifferential. The subdifferential can be thought of as the set of all possible tangents to the tax function. We refrain from further details here.

Looking at the mathematical conditions under which joint filing leads to a lower overall tax burden, one again encounters a convex tax function. Let b_1 and b_2 denote the taxable incomes of the two spouses. If $\mathrm{Tax}(\cdot)$ denotes the tax liability function, then under separate assessment their combined liability is $\mathrm{Tax}(b_1) + \mathrm{Tax}(b_2)$. Under joint filing, however, the spouses' incomes are pooled and split evenly, yielding a tax liability of $2\mathrm{Tax}\left(\frac{b_1+b_2}{2}\right)$. If joint filing reduces the overall burden, this requires that

$$\mathrm{Tax}(b_1) + \mathrm{Tax}(b_2) < 2\,\mathrm{Tax}\left(\frac{b_1 + b_2}{2}\right) , \tag{3.5}$$

regardless of the individual income levels. This condition is precisely the requirement that $\mathrm{Tax}(\cdot)$ be (strictly) convex.[14]

For reasons of convenience, Becker and Löffler also assume that the tax function does not change over time. This restriction is less severe than it might appear, since otherwise the function could simply be indexed by time t and all previous considerations would still apply. More critical, however, is the assumption that the future tax function is deterministic—that is, that one already knows today how much tax will be owed in the future for a given tax base. Yet, as we have already noted, there is scarcely any literature addressing stochastic tax rates, let alone stochastic tax schedules.

When moving from a linear to a convex tax liability function, the effects of nonlinear taxation can be clearly illustrated. As in Becker and Löffler, we rely only on the principle of absence of arbitrage, which we have already discussed extensively elsewhere. We now ask which firm values are admissible under arbitrage-free conditions. The key results are Becker and Löffler (2024, Theorems 8 and 9), which we incorporate into our model, using the notation introduced earlier.

At this point, it is useful to change our approach. Until now, we have assumed that both a firm's cash flows and its values are given, and that these quantities are related in such a way that no arbitrage opportunities arise. Becker and Löffler also start from given cash flows CF_t, but they do not assume that the corresponding values V_t are already known. Instead, their aim is to clarify which values V_t are even *conceivable* under no-arbitrage conditions.

Suppose, for instance, that someone were to assign arbitrary numbers—say, numbers determined by the roll of a die—as "values" V_t to a firm's given cash flows CF_t. In such a case, arbitrage opportunities would materialize, even in a no-tax setting. The situation should be no different once nonlinear income taxation is taken into account. It is therefore natural to ask whether, among all conceivable numbers, there are some that qualify as prices precisely because they do not permit arbitrage opportunities.

It is well established that under a linear tax rate, the Fundamental Theorem (Thm. 3.2) takes the following form:

$$V_t = \frac{CF_{t+1} - \tau B_t}{1 + r_f(1-\tau)} . \tag{3.6}$$

[14] Convexity is usually defined somewhat differently: instead of the coefficients $\frac{1}{2}$, one uses arbitrary positive real numbers that sum to one. The formulation employed here is, interestingly, taken directly from Jensen's original article in which he was the first to mathematically analyze the concept of convexity.

Here, τ denotes the tax rate, assumed to be constant over time and independent of the tax base. In the nonlinear case, however, Becker and Löffler take the opposite route. For any conceivable price V_t, they calculate the linear tax rate τ that would generate exactly that price. To do so, they rearrange Eq. (3.6) with respect to τ and solve. As long as $B_t \neq r_f V_t$, this procedure always works. The resulting number is what Becker and Löffler call the "implicit tax rate."

It is worth noting that the computed τ is not necessarily a value one would ordinarily regard as a tax rate. It may well exceed 100% or even be negative. What, then, is the meaning of this τ? The implicit tax rate makes it perfectly transparent whether or not arbitrage opportunities exist. To dig deeper into this point, we must introduce a variable that has so far played no role in our valuation theory: the initial endowment of investors.

In our earlier use of DCF, the size of the investor's wealth—whether buying or selling a firm—was irrelevant. Since the purchase price could be determined without reference to it, we simply ignored the investor's overall wealth. The same holds, incidentally, when a linear income tax is incorporated into the valuation.

Things change, however, once nonlinear taxes come into play. Whether an arbitrage opportunity exists may now depend on the size of the investor's assets outside the firm being valued but nonetheless subject to taxation. This should not be surprising: the magnitude of these other assets directly affects the marginal tax rate!

Thus, under nonlinear taxation, we must take into account the taxable income the investor receives in addition to that from the firm under consideration. In what follows, we refer to these additional sources of income as the "initial endowment".

Becker and Löffler (2024) established the following results:

CASE 1 Suppose implicit tax rates can be evaluated (recall that this is possible whenever $B_t \neq rV_t$). In this case, it becomes essential to differentiate between two distinct subcases.

SUBCASE 1.1 If these implicit tax rates coincide with the marginal tax rates applied to the income from the investor's initial endowment, then the market is free of arbitrage.

SUBCASE 1.2 Otherwise there are arbitrage opportunities.[15]

In other words, if we look only at the investor's initial endowment and set the firm under valuation aside, there is a characteristic marginal tax rate attached to it. When firm values are then computed from the company's cash flows, that same marginal tax rate must be applied. Otherwise, an arbitrage opportunity arises.

CASE 2 If no implicit tax rates can be evaluated (this is the case if $B_t = rV_t$), the only arbitrage-free prices are those that would prevail in a world without taxes.[16]

All of these considerations matter because, taken together, they lead to a simple conclusion: under a nonlinear tax, Eq. (3.6) still applies. The only change lies in the

[15] It may well occur—indeed, this is a characteristic feature of nonlinear taxation—that such arbitrage opportunities are bounded, in the sense that repeated exploitation does not yield unbounded gains. For a detailed discussion, see Becker and Löffler (2024).

[16] This case is of particular interest in the tax literature, as it describes the so-called investment-neutral taxes.

tax rate used. It is now a marginal tax rate, determined either by the investor's initial endowment or—if none exists—reduced to zero. In this respect, and only in this respect, the treatment of nonlinear taxation is more complicated than that of a linear income tax.

Let us close with a final remark that goes beyond our discounted cash flow framework. In this book, we have argued that arbitrage theory can stand independently alongside equilibrium theory, precisely because it rests on fewer assumptions. The work of Becker and Löffler (2024) offers further support for this claim.

To see why, consider how equilibrium theory would need to proceed in order to analyze the effects of nonlinear taxation. The first step would be to describe the equilibrium. Naturally, it would involve several individuals with different initial endowments—otherwise, the nonlinearity of the tax would never come into play. But once this assumption is taken seriously, a logical contradiction emerges.

Consider two individuals whose endowments differ and who therefore face different marginal tax rates. Following the logic of Becker and Löffler (2024), each individual would first ask which prices rule out arbitrage. Whether supply and demand balance to produce an equilibrium can be set aside. The key problem is this: both individuals apply Eq. (3.6), but each inserts a different tax rate. The prices they compute—prices meant to rule out arbitrage—therefore differ.

But that, of course, cannot happen in a functioning market. Prices in such markets are always unique. It is not possible for one individual to acquire a security or a good at price p, while another pays a different price p' for the very same asset.

What does this insight tell us? The only possible conclusion is that equilibrium models cannot accommodate nonlinear taxes *and* heterogeneous endowments simultaneously without running into logical contradiction. Together, these features make a unified modeling approach impossible. Anyone wishing to study the effects of nonlinear income taxation must therefore turn to a different framework. This conclusion may well come as a surprise.

3.1.6 Tax Shield and Distribution Policy

In this section we want to characterize the difference in value between an unlevered and a levered firm.

Let us begin with the firm with full distribution. From Thm. 3.1 together with (3.3), we immediately get the representation

$$\widetilde{V}_t^u = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_T^u\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}} \,.$$

We get the value of a levered firm in the exact same way from

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^l\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_T^l\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}} \,.$$

Yet, we still have to think about how the free cash flows from the firm with partial distribution differ from those of the unlevered firm. We thereby adopt the following principle: The first earnings retention takes place in t. It is economically unsuitable to forgo with distributions at the last time $t = T$. From that results $\widetilde{A}_T = 0$. If we compare the value of the levered and unlevered firm, we then obtain, by applying Rules 4 and 5

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{\left(1-\tau^D\right) \mathrm{E}_t^Q\left[\left(1+r_f\right)\widetilde{A}_t - \widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\left(1-\tau^D\right) \mathrm{E}_t^Q\left[\left(1+r_f\right)\widetilde{A}_{T-2} - \widetilde{A}_{T-1}\right]}{1+r_f\left(1-\tau^I\right)^{T-t-1}} + \frac{\left(1-\tau^D\right) \mathrm{E}_t^Q\left[\left(1+r_f\right)\widetilde{A}_{T-1}\right]}{(1+r_f(1-\tau^I))^{T-t}} .$$

After some minimal reshuffling, the following results

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{\left(1-\tau^D\right)\left(1+r_f\right)\widetilde{A}_t}{1+r_f\left(1-\tau^I\right)} + \frac{\mathrm{E}_t^Q\left[\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_{t+1} - \left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\mathrm{E}_t^Q\left[\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_{T-1} - \left(1-\tau^D\right)\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}} .$$

This brings us to the conclusion

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \left(1-\tau^D\right)\widetilde{A}_t + \frac{\tau^I\left(1-\tau^D\right) r_f \mathrm{E}_t^Q\left[\widetilde{A}_t\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\tau^I\left(1-\tau^D\right) r_f \mathrm{E}_t^Q\left[\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}} . \quad (3.7)$$

This equations shows itself to be the personal income tax pendant to Eq. (2.14). In place of debt $\widetilde{D}_t$, $\left(1-\tau^D\right)\widetilde{A}_t$ simply enters in, that being the amount by which the maximum distribution to the financiers is reduced.

Alternative Retained Earnings Policies If we push the logic of tax shields to its conclusion, we would then have to advise every firm to defer distributions as long as possible for tax reasons. There is nothing to object to this recommendation within the framework of our model. We could have also argued accordingly in the last chapter. It was the debt which brought a tax shield there. And it would have then made sense to recommend that the firm allow for the maximum debt ratio permissible. As we did there, we here refrain from such recommendations since we very well know that suitable advice is unreasonable if it is solely based on tax considerations. That is also why we want to master the situation differently here. We take the firm's distribution policy—just as we did the debt policy in the last chapter—as a given and question how it affects the value of the firm.

In what follows, we analyze five payout policies more precisely. Because payout and retention are complementary, we can equally frame them as alternative retention policies. We adopt labels for the subsequent characterizations that are easy to remember.

1. Under *autonomous retention*, a deterministic amount is retained each period.
2. Under *cash flow based earnings retention*, a fixed fraction of each period's free cash flow is retained.
3. Under *dividend based earnings retention*, retention is chosen so that a fixed dividend is paid in each of the first *n* periods.
4. Under *market value based earnings retention*, the deducted amounts are so chosen that the relation of earnings retention and equity value remains deterministic.

Another dividend policy becomes relevant under statutory payout restrictions (disbursement stoppages). In many jurisdictions, firms may distribute at most their current earnings, even when free cash flow is higher. Because such restrictions are difficult to model, we will not pursue them further.

3.1.7 Examples and Problems

3.1.7.1 The Finite Case

We also reuse the numbers from the two examples in the previous chapter, now in the context of personal income taxation. We assume that the tax rates on dividends and on interest coincide and denote the common rate by τ. Because the derivations in this chapter are formally almost identical to those in the previous one, we could, in principle, repeat the same calculations, changing only the interpretation of the variables: With $\widetilde{CF}_t^u$ denoting after-income-tax cash flows, $k^{E,u}$ the after-tax cost of equity, and τ the income-tax rate, the calculations for both the finite and infinite examples proceed exactly as in Sect. 1.3.3 and 1.4.6; only the interpretation of the variables changes.

In the infinite case, despite the formal parallel with the previous chapter, we would stumble into a pitfall. Analogous to Sect. 1.4.6, we might compute risk-neutral probabilities $Q(d)$ and $Q(u)$ for a given period. The resulting values no longer match those obtained for the infinite case in Sect. 1.4.6, because the fundamental theorems differ between corporate and personal taxation. In the former, one discounts at the risk-free rate r_f; in the latter, at the after-tax risk-free rate $r_f(1-\tau)$. For the parameter constellation we chose, this switch can drive the implied probabilities negative—an immediate red flag for arbitrage.[17]

We therefore assume a cost of equity of $k^{E,u} = 15\%$. At this rate, the arbitrage opportunity disappears. With $k^{E,u} = 15\%$, the value of the unlevered firm in the infinite example is

$$V_0^u = \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}} + \frac{\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1+k^{E,u})^2} + \frac{\mathrm{E}\left[\widetilde{CF}_3^u\right]}{(1+k^{E,u})^3}$$

[17] For details see Prob. 3.1.

$$= \frac{100}{1.15} + \frac{110}{1.15^2} + \frac{121}{1.15^3} \approx 249.69 \ .$$

The values $\widetilde{V}_1^u$ and $\widetilde{V}_2^u$ must be recomputed.

All other former numerical values for the infinite example carry over unchanged.

3.1.7.2 Problems

Problem 3.1 In Sect. 1.4.6 we were able to evaluate the risk-neutral probabilities $Q_1(d)$ and $Q_1(u)$ for the finite example. Show that

$$Q_1(u) \approx -0.125, \qquad Q_1(d) \approx 1.125$$

if a personal income tax with $\tau = 50\%$ is present.

Verify that for $k^{E,u} = 15\%$ this arbitrage opportunity vanishes and determine $Q_1(u)$ and $Q_1(d)$. Determine $Q_2(dd), Q_2(du), Q_2(ud)$ and $Q_2(uu)$.

Problem 3.2 Prove that the tax shield $\widetilde{V}_t^l - \widetilde{V}_t^u$ in the case of the personal income tax satisfies

$$\widetilde{V}_t^l - \widetilde{V}_t^u = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l - \widetilde{V}_{t+1}^u\right]}{1 + r_f(1-\tau^I)} + (1-\tau^D)\frac{\mathrm{E}_t^Q\left[(1+r_f)\widetilde{A}_t - \widetilde{A}_{t+1}\right]}{1 + r_f(1-\tau^I)} \ .$$

Problem 3.3 Similar to Prob.2.8 show that the main valuation Eq. (3.7) can be written as

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{1-\tau^D}{1-\tau^I}\widetilde{A}_t + \frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s - \widetilde{A}_{s-1}\right]}{\left(1 + r_f\left(1-\tau^I\right)\right)^{s-t}} \ .$$

3.2 Cost of Equity and Varying Tax Rate

3.2.1 A Convincing but Misleading Idea

Statement of the Problem In this section, we address a question we have so far set aside: How do a firm's cost of equity change when the income tax rate changes? And, relatedly, how does the value of a fully self-financed firm depend on the income tax rate? In all previous discussions we held tax rates constant and did not analyze how the firm's market value responds to changes in taxation. The reason for this omission will become clear below.

Anyone who wants to investigate the relation between the tax rate and the value of the firm must grapple with two problems. On the one hand, one must analyze the effect of the tax rate on cash flows; on the other hand, the effect on the cost of equity. The first problem is generally considered straightforward in the literature and therefore receives little attention. By contrast, how the cost of equity reacts to changes in the tax rate deserves much more consideration. In what follows, we restrict attention to the case of a perpetuity. For simplicity, we assume that both types of income are taxed identically; the tax rate is denoted by τ.

Influence on Cash Flows We assume that the expected cash flows are constant. We further assume that there are no capital contributions or withdrawals and no additional investments over the horizon. Hence, the free cash flow equals the gross cash flow multiplied by $1-\tau$,

$$\widetilde{CF}_t^u = \widetilde{GCF}_t(1-\tau) \;. \tag{3.8}$$

Influence on Cost of Equity Following Johansson (1969), the literature often posits a functional relationship between the post-tax cost of equity and the tax rate,[18] which can be formulated as follows: the post-tax cost of equity $k^{E,u}$ depends linearly on the tax rate τ; that is, there exists a constant k^E such that

$$k^{E,u} = k^E(1-\tau) \;. \tag{3.9}$$

The constant k^E is often called the "pre-tax cost of equity," although this interpretation is not needed for our argument. We have deliberately refrained from asserting such a relation.

Eq. (3.8), in conjunction with Eq. (3.9), yields the familiar result that, under the assumptions made here, income tax is redundant: it cancels out of the valuation equation,[19]

$$\widetilde{V}_t = \frac{\widetilde{CF}_t^u}{k^{E,u}} = \frac{\widetilde{GCF}_t(1-\tau)}{k^E(1-\tau)} = \frac{\widetilde{GCF}_t}{k^E} \;. \tag{3.10}$$

But our discussion concerns a deeper matter. We want to critically analyze the linear relation (3.9).

Stochastic Structure of the Cash Flows There is a strong relation between the cost of equity and the risk-neutral probability measure Q. We always stressed that earlier as well. The relation we wish to emphasize here is particularly well captured by Theorems 2.3 and 3.4. If, as now in Eq. (3.9), a statement on the dependance of the cost of equity on the tax rate is made, then that also implies a relation between risk-neutral probabilities and the tax rate.

[18] Johansson (1969) examined—more generally—whether the formula stated below is actually applicable and emphasized that several conditions must be satisfied. Consequently, this relation is typically presented in examples rather than in theoretical chapters; see, e.g., Hillier et al. (2008, Example 14.5, p. 520) or practice-oriented texts such as Institut der Wirtschaftsprüfer in Deutschland (2008, Sect.4.4.2.5) (in German).

[19] There has been a heated debate in the German literature about this relation, and it has found its way into the documents of the Institute of Public Auditors in Germany ("Institut der Wirtschaftsprüfer", IdW). For example, Institut der Wirtschaftsprüfer in Deutschland (2013, II. Band, Teil A, Rz. 195) notes: "... in 1983 it was still assumed that in a number of cases one could forgo the (explicit) inclusion of the investor's tax burden, since it has no effect on the firm's value." The Institute, however, abandoned this position in 1997. Since then, an income tax rate of 35% has been used if no other rate can be identified; see Institut der Wirtschaftsprüfer in Deutschland (2000) and Institut der Wirtschaftsprüfer in Deutschland (2013, II. Band, Teil A, Rz. 117).

We use an example to show that this relation can lead to a serious problem. To characterize the stochastic structure of future gross cash flows more precisely, we return to our infinitely lived firm. We assume that the firm's gross cash flows, $\widetilde{GCF}_t$, follow the binomial process shown in Fig. 1.3.

In order to show that the free cash flows as modeled are in fact martingale-like, we have to make it clear how the conditioned expectation $\mathrm{E}_t\left[\cdot\right]$ is calculated. At time t the cash flow $\widetilde{GCF}_t$ is already known, and that is why the uncertainty can only relate the subsequent movement u or d. We thus have, in connection with Rule 2

$$\begin{aligned}
\mathrm{E}_t\left[(1-\tau)\,\widetilde{GCF}_{t+1}\right] &= (1-\tau)\,\mathrm{E}_t\left[\widetilde{GCF}_{t+1}\right] \\
&= (1-\tau)\,P(u)\,u\,\widetilde{GCF}_t + (1-\tau)\,P(d)\,d\,\widetilde{GCF}_t \\
&= \Big(\underbrace{P(u)u + P(d)d}_{:=1+g}\Big)(1-\tau)\,\widetilde{GCF}_t\ . \qquad (3.11)
\end{aligned}$$

Because of (3.8) that is exactly the assumption of martingale-like cash flows.

An Arbitrage Opportunity Now we do not only suppose the existence of one, but of two firms. Both should be without debt and pursue a policy of full distribution. For the parameters u, d in the first firm

$$P(u)u + P(d)d = 1 \qquad \Longrightarrow \qquad g = 0$$

should be valid. The value of the firm at time t is denoted by $\widetilde{V}_t$. For the sake of simplicity, we designate the cost of equity when taxes are neglected with k; they should remain constant in time.

The second firm should also possess gross cash flows with the stochastic structure as in Fig. 1.3. If the cash flows grow in the first firm (that is, move up), then they also grow in the second firm. If they fall in the first firm, then they also decline in the second. It can thus be determined that the cash flows of the two firms are perfectly correlated. we will denote the cash flows of the second firm with $\widetilde{GCF}'_t$. The factors u', d' are different from those of the first firm, but

$$P(u)u' + P(d)d' = 1 \qquad \Longrightarrow \qquad g' = 0$$

should again be valid. Because of this connection, the gross cash flows do not point to any expected growth in either case. The second firm's cost of equity rate when taxes are neglected is k' and the firm's value in t is denoted with $\widetilde{V}'_t$.

The investor may continue to buy or sell risk-free bonds whose value at time t is B_t. Ignoring taxes, these bonds earn the risk-free rate r_f.

We use a well-known idea in the literature, pricing by duplication. In this way we can derive a relationship between the values of the two firms, $\widetilde{V}_t$ and $\widetilde{V}'_t$, and likewise between their costs of equity, k and k'. The idea is that one can construct a portfolio from shares of the first firm and a risk-free bond whose cash flows replicate those available to an owner of the second firm.

For this purpose, we construct a portfolio that at time t contains exactly n_B risk-free bonds and n_V shares of the first firm. We choose n_B and n_V so that, regardless of the state that realizes at time $t+1$, the following

$$n_B B_t \left(1 + r_f(1-\tau)\right) + n_V \left(\widetilde{GCF}_{t+1}(1-\tau) + \widetilde{V}_{t+1}\right) = \widetilde{GCF}'_{t+1}(1-\tau) + \widetilde{V}'_{t+1}$$

is satisfied. With the help of (3.10), applied to both firms, the equation can be simplified to

$$n_B B_t \left(1 + r_f(1-\tau)\right) + n_V \left(1 + k_{t+1}(1-\tau)\right) \widetilde{V}_{t+1} = \left(1 + k'_{t+1}(1-\tau)\right) \widetilde{V}'_{t+1} \ .$$

In the period following from the end of time t, there are exactly two possible directions (up or down) along the cash flow path in the binomial model. That is why the above condition can be resolved in a system of two equations, which must be simultaneously satisfied: in the case of an up movement

$$\left(1 + r_f(1-\tau)\right) n_B B_t + u \left(1 + k(1-\tau)\right) n_V \widetilde{V}_t = u' \left(1 + k'(1-\tau)\right) \widetilde{V}'_t$$

must be valid, while in the case of a down movement

$$\left(1 + r_f(1-\tau)\right) n_B B_t + d \left(1 + k(1-\tau)\right) n_V \widetilde{V}_t = d' \left(1 + k'(1-\tau)\right) \widetilde{V}'_t$$

must be satisfied. These two equations form a linear system that can be solved uniquely for n_B and n_V,

$$\begin{aligned} n_B &:= \frac{\widetilde{V}'_t}{B_t} \frac{(u-u')(1+k'(1-\tau))}{u(1+r_f(1-\tau))} \\ n_V &:= \frac{\widetilde{V}'_t}{\widetilde{V}_t} \frac{u'(1+k'(1-\tau))}{u(1+k(1-\tau))} \ . \end{aligned}$$

All variables are stochastic. They depend upon the firm value in t.

Since, by construction, the portfolio delivers the same cash flows at time $t+1$ as the second firm, it must—under arbitrage-free conditions—have the same price at time t,

$$n_B B_t + n_V \widetilde{V}_t = \widetilde{V}'_t \ . \tag{3.12}$$

If we substitute the solutions for n_B and n_V into the preceding equation, we obtain a valuation equation for the second firm. In doing so, $\widetilde{V}'_t$ cancels, yielding a functional relation between the cost of equity k of the first firm and k' of the second,

$$\frac{u-u'}{1+r_f(1-\tau)} + \frac{u'}{1+k(1-\tau)} = \frac{u}{1+k'(1-\tau)} \ . \tag{3.13}$$

A possible economic interpretation of this equation could consist of the cost of equity $1 + k'(1-\tau)$ being established as harmonic mean of the cost of equity $1 + r_f(1-\tau)$ and $1 + k(1-\tau)$, whereby this harmonic mean is weighted with the parameters of the up and down movement.

The following is decisive for this equation: it must be valid for all conceivable tax rates τ. But that does not work. In addition to the trivial solution $\tau = 1$ the value $\tau = 0$ will yield a relation between k and k'. Hence, $\tau = 0$ and $\tau = 1$ solve the above equation already. But, a simple rearrangement shows that (3.13) is a quadratic equation in τ that cannot have more than two solutions! The equation cannot be satisfied for a single further τ.

That violates the no-arbitrage principle, which we always uphold. If the cost of equity does not satisfy relation (3.13), then the corresponding relation for firm values (3.12) also fails to hold—allowing an immediate arbitrage. Depending on whether

$$n_B B_t + n_V \widetilde{V}_t > \widetilde{V}'_t \qquad \text{or} \qquad n_B B_t + n_V \widetilde{V}_t < \widetilde{V}'_t$$

is valid, you must either go short or long with the shares of the second firm and cover this transaction with the bond and the shares of the first firm.

We asked what connection exists between the cost of equity and the tax rate—a question we had set aside until now. To address it, we drew on a concept popular in applied work that posits a simple linear link between the two. We showed that unrestricted use of the resulting equation can generate arbitrage. The upshot is clear: anyone seeking to understand how the cost of equity responds to changes in the tax rate should not rely on Eq. (3.9). DCF theory does not provide an answer here—which is why we have deliberately avoided the question until now.

3.2.2 Problems

The following problems are devoted to the understanding of the arbitrage opportunity revealed in this section.

Problem 3.4 One particular feature of our tax system in that section was that only dividends were taxed. Assume now that also capital gains are taxed. In particular, we assume that the capital gains (even if they are not realized!) also add to the tax base, i.e., instead of (3.8) we assume

$$\widetilde{CF}^u_t = \widetilde{GCF}_t - \tau\Big(\widetilde{GCF}_t + \underbrace{\widetilde{V}^u_t - \widetilde{V}^u_{t-1}}_{\text{unrealized capital gain}}\Big) .$$

Such a tax system is also called neutral tax system or taxation of economic rent. Show that if the value of the assets remain unchanged by the tax rate the cost of equity have to satisfy

$$k^{\text{post-tax}} = k^{\text{pre-tax}}(1 - \tau) .$$

Remark: It can be shown that if this system is free of arbitrage pre-tax it will remain free of arbitrage post-tax.

Problem 3.5 In this problem, we derive a relation between firm value and the tax rate that does not violate the arbitrage principle.

Assume that the risk-neutral probability measure Q does not change with the tax rate τ and that the cash flows form a perpetual rent (no growth). We further assume that the company has constant pre-tax cost of equity $k^{\text{pre-tax}}$. Using Thm. 3.2, derive the following equation for post-tax value of the unlevered firm that explicitly contains $k^{\text{pre-tax}}$:

$$\widetilde{V}_t^u = \frac{(1-\tau)\widetilde{GCF}_t}{(1+k^{\text{pre-tax}})\frac{1+r_f(1-\tau)}{1+r_f} - 1}$$

Hint: The heart of the solution is the precise definition of a pre-tax cost of equity. Make sure that

$$\frac{\mathrm{E}_t^Q\left[\widetilde{GCF}_s\right]}{\left(1+r_f\right)^{s-t}} = \frac{\mathrm{E}_t\left[\widetilde{GCF}_s\right]}{(1+k^{\text{pre-tax}})^{s-t}}$$

is a good choice.

Problem 3.6 The (pre-tax) gross cash flows from two companies follow the binomial tree as in Fig. 1.3 and let $CF_0 = 100$, $g = 0$, $r_f = 5\%$, $k = 15\%$, $u = 10\%$ and $u' = 20\%$.

a) Consider the first company having cost of equity k. Determine the risk-neutral probabilities $Q(u)$ and $Q(d)$.
b) Use the arbitrage argument above to determine k'.
c) Assume that for the first company the post-tax cost of equity are given as $(1-\tau)k$. Determine the risk-neutral probabilities depending on τ.
d) Now assume that, for the second company, the post-tax cost of equity is given by $(1-\tau)k'$. Calculate again the risk-neutral probabilities depending on τ. Does the result coincide with c)?

Hint: You might consult the finite example from Sect. 2.1.5 on how to calculate $Q(u)$ and $Q(d)$.

3.3 Retention Policies

In what follows, we analyze alternative earnings-retention policies. With one exception, we focus on strategies for which the retention rate is stochastic.

3.3.1 Autonomous Retention

The free cash flow of an unlevered firm may be distributed in full or only in part. We now consider the simplest form of a retention policy, characterized by the firm's practice of withholding, as a matter of principle, a fixed amount from the maximum distributable cash flow each year.

Definition 3.2 (Autonomous retention) *A firm is following autonomous retention policy if the retention is deterministic.*

The value of a firm which follows this policy is straightforward to calculate. We employ Definition 3.2 in Eq. (3.7) and get

Theorem 3.5 (Autonomous retention) *In the case of an autonomous retention, the following is valid*

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \left(1 - \tau^D\right) A_t + \frac{\tau^I \left(1 - \tau^D\right) r_f A_t}{1 + r_f \left(1 - \tau^I\right)} + \ldots + \frac{\tau^I \left(1 - \tau^D\right) r_f A_{T-1}}{\left(1 + r_f \left(1 - \tau^I\right)\right)^{T-t}} \,.$$

Perpetual Firm with an Autonomous Retention If the firm pursues the autonomous retention in such a way so that A_t does not change in time, the last statement is simplified. We furthermore suppose at the same time that the firm lives infinitely. Notice that the value of the tax shield in Thm. 3.6 does not depend on the tax rate if $\tau^D = \tau^I$.

Theorem 3.6 (Constant retention) *The firm lives on forever. For constant A, the following applies*

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{1 - \tau^D}{1 - \tau^I} A \,.$$

The proof of this statement is straightforward.

3.3.2 Example

3.3.2.1 The Finite Example

Let us turn to our finite example. The amounts of retention with the levered firm are exactly

$$A_0 = 10, \quad A_1 = 20, \quad A_2 = 0 \,.$$

With that we get

$$\begin{aligned} V_0^l &= V_0^u + (1-\tau)A_0 + \frac{\tau(1-\tau)r_f A_0}{1+r_f(1-\tau)} + \frac{\tau(1-\tau)r_f A_1}{(1+r_f(1-\tau))^2} \\ &= 249.692 + (1-0.5)\times 10 + \frac{0.5\times(1-0.5)\times 0.1\times 10}{1+0.1\times(1-0.5)} + \frac{0.5\times(1-0.5)\times 0.1\times 20}{(1+0.1\times(1-0.5))^2} \\ &\approx 255.38 \end{aligned}$$

for the value of the levered firm.

In order to establish the value of the levered firm in the infinite example, assuming $A = 10$ and using the statement from Thm. 3.6 we get

$$V_0^l = V_0^u + A = 510\,,$$

which is independent of the tax rate as mentioned earlier.

3.3.3 Retention Based on Cash Flow

The next policy concerns the case where a fraction of the distributable cash flow is retained.

Definition 3.3 (Retention based on cash flows) *A firm is following an earnings retention policy based on cash flows if the retention is a determinate multiple of the free cash flow of the unlevered firm,*

$$\widetilde{A}_t = \alpha_t\, \widetilde{CF}_t^u\,.$$

α_t is a number greater than zero.

The value of a firm which follows this policy is easy to calculate. We employ Definition 3.3 in Eq. (3.7) and get

$$\begin{aligned} \widetilde{V}_t^l &= \widetilde{V}_t^u + \left(1-\tau^D\right)\widetilde{A}_t + \frac{\mathrm{E}_t^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_t\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\mathrm{E}_t^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}} \\ &= \widetilde{V}_t^u + \left(1-\tau^D\right)\alpha_t\,\widetilde{CF}_t^u + \frac{\tau^I r_f\left(1-\tau^D\right)\alpha_t\,\widetilde{CF}_t^u}{1+r_f\left(1-\tau^I\right)} \\ &\quad + \frac{\tau^I r_f\left(1-\tau^D\right)}{1+r_f\left(1-\tau^I\right)}\left(\frac{\mathrm{E}_t^Q\left[\alpha_{t+1}\widetilde{CF}_{t+1}^u\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\mathrm{E}_t^Q\left[\alpha_{T-1}\widetilde{CF}_{T-1}^u\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t-1}}\right). \end{aligned}$$

All we need to do now is to use Thm. 3.4, and the following statement is already proven.

Theorem 3.7 (Cash flow-retention) *In the case of a retention based on cash flows, the following is valid*

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{(1+r_f)\left(1-\tau^D\right)\alpha_t \widetilde{CF}_t^u}{1+r_f\left(1-\tau^I\right)} + \frac{\tau^I r_f\left(1-\tau^D\right)}{1+r_f\left(1-\tau^I\right)}\left(\frac{\mathrm{E}_t\left[\alpha_{t+1}\widetilde{CF}_{t+1}^u\right]}{1+k^{E,u}} + \ldots + \frac{\mathrm{E}_t\left[\alpha_{T-1}\widetilde{CF}_{T-1}^u\right]}{(1+k^{E,u})^{T-t-1}}\right).$$

Perpetual Firm with Constant Retention If the firm follows a cash-flow–based retention rule such that the retained share $\alpha > 0$ is constant over time, the preceding statement simplifies. This simplification, however, requires an infinite horizon: with a finite lifetime one must have $\alpha_T = 0$ at the terminal date, so a constant retention rate is no longer meaningful. The following result is stated under these assumptions.

Theorem 3.8 (Constant retention rate) *The firm lives on forever. For constant α, the following applies*

$$\widetilde{V}_t^l = \left(1 + \frac{\tau^I r_f\left(1-\tau^D\right)\alpha}{1+r_f\left(1-\tau^I\right)}\right)\widetilde{V}_t^u + \frac{(1+r_f)\left(1-\tau^D\right)\alpha}{1+r_f\left(1-\tau^I\right)}\widetilde{CF}_t^u .$$

The proof of this statement is simple. We only need to employ the constant retention rate,

$$\begin{aligned}\widetilde{V}_t^l &= \widetilde{V}_t^u + \frac{(1+r_f)\left(1-\tau^D\right)\alpha\widetilde{CF}_t^u}{1+r_f\left(1-\tau^I\right)} + \frac{\tau^I r_f\left(1-\tau^D\right)\alpha}{1+r_f\left(1-\tau^I\right)}\sum_{s=t+1}^{\infty}\frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{(1+k^{E,u})^{s-t}} \\ &= \widetilde{V}_t^u + \frac{(1+r_f)\left(1-\tau^D\right)\alpha\widetilde{CF}_t^u}{1+r_f\left(1-\tau^I\right)} + \frac{\tau^I r_f\left(1-\tau^D\right)\alpha}{1+r_f\left(1-\tau^I\right)}\widetilde{V}_t^u .\end{aligned}$$

And that was it. □

3.3.4 Example

Let us turn again to our finite example. The retention coefficients with the levered firm are exactly

$$\alpha_0 = 0\%, \quad \alpha_1 = 10\%\,;, \quad \alpha_2 = 20\%\,.$$

With that we get

$$\begin{aligned} V_0^l &= V_0^u + \frac{\left(1+r_f\right)(1-\tau)\alpha_0 CF_0^u}{1+r_f(1-\tau)} + \frac{\tau r_f(1-\tau)}{1+r_f(1-\tau)}\left(\alpha_1 \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}}\right. \\ &\quad \left. +\alpha_2 \frac{\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1+k^{E,u})^2}\right) \\ &= 249.692 + 0 + \frac{0.5\times 0.1\times(1-0.5)}{1+0.1\times(1-0.5)} \times \left(0.1\times\frac{100}{1+0.15} + 0.2\times\frac{110}{(1+0.15)^2}\right) \\ &\approx 250.29 \end{aligned}$$

for the value of the levered firm.

In order to establish the value of the levered firm in the infinite example, we use the statement from Thm. 3.8 and get

$$\begin{aligned} V_0^l &= \left(1+\frac{\tau r_f(1-\tau)}{1+r_f(1-\tau)}\,\alpha\right)V_0^u + \frac{\left(1+r_f\right)(1-\tau)\alpha}{1+r_f(1-\tau)}\,CF_0^u \\ &= \left(1+\frac{0.5\times 0.1\times(1-0.5)}{1+0.1\times(1-0.5)}\times 0.5\right)\times 500 + \frac{(1+0.1)\times(1-0.5)\times 0.5}{1+0.1\times(1-0.5)}\times 100 \\ &\approx 558.33\,. \end{aligned}$$

3.3.5 Retention Based on Dividends

Valuation Equation We now want to look at a retention policy in which a prescribed (pre-tax) dividend is distributed for a period of n years. At the end of this period, the firm should switch to a policy of full distribution. In the following, we assume that all decisions about the firm's investment policy have already been made. However, the firm cannot pay dividends if the free cash flow is not large enough. It is thus recommended to require that the free cash flow be higher, or at worst equal to, the planned dividend. If the two amounts do not match, the difference is withheld. Repayments of this difference are then available as additional distribution potential at time $t+1$.

Definition 3.4 (Retention based on dividends) *A firm is following a retention policy based on dividends if a prescribed deterministic pre-tax dividend should be payed in the first $n \le T$ periods,*

$$\widetilde{A}_t = \left(\frac{1}{1-\tau^D}\,\widetilde{CF}_t^u + (1+r_{t-1})\widetilde{A}_{t-1} - \,Div_t\right)^+$$

for all $t = 1, \ldots, n$.

Readers who still remember details from Sect. 2.7.1 and 2.7.3 of the previous chapter, will hardly be surprised by the assertion that a general valuation equation is only possible through the inclusion of derivatives which we believe are not always traded in the market. Such complications can only be avoided if we introduce an additional assumption.

Assumption 3.2 (Non-negative retention) *At no time does the prescribed dividend policy lead to the retention amount being negative,*

$$\frac{1}{1-\tau^D}\widetilde{CF}_t^u \geq Div_t \qquad \forall t \leq n \;.$$

The maximum function in Def. 3.4 is superfluous under this simplified assumption, and we can prove that the following statement is valid.

Theorem 3.9 (Retention based on dividends) *The following applies in the case of a retention based on dividends, if the dividend is not greater than the free cash flow of the unlevered firm for all times before* T*,*

$$\widetilde{V}_u^l = \widetilde{V}_t^u + \tau^I\left(1-\tau^D\right)\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n-t+1}\widetilde{A}_t +$$
$$+\tau^I r_f \sum_{v=t+1}^{n}\left(\frac{\mathrm{E}_t\left[\widetilde{CF}_v^u\right]}{(1+k^{E,u})^{v-t}} - \frac{(1-\tau^D)Div_v}{\left(1+r_f\left(1-\tau^I\right)\right)^{v-t}}\right)\left(1+\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n+1-v}\right) \;.$$

The proof is found in the appendix. In our opinion it does not make sense to generalize the above statement to the case of an infinitely lived firm with a perpetual constant dividend ($n \longrightarrow \infty$), since we otherwise fall into conflict with the assumption of transversality. We already pointed this out in a similar context in Sect. 2.7.3.

3.3.6 Example

Let us work on the basis that the firm will pay the following pre-tax dividend:

$$A_0 = 0, \quad Div_1 = Div_2 = 40 \;.$$

If the tax rate amounts to 50% , Assump. 3.2 is satisfied. Under these conditions we get the following for the value of the levered firm

$$V_0^l = V_0^u + \tau r_f \left(\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}} - \frac{(1-\tau)Div_1}{1+(1-\tau)r_f} \right) \left(1 + \left(\frac{1+r_f}{1+(1-\tau)r_f} \right)^2 \right)$$
$$+ \tau r_f \left(\frac{\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1+k^{E,u})^2} - \frac{(1-\tau)Div_2}{(1+(1-\tau)r_f)^2} \right) \left(1 + \frac{1+r_f}{1+(1-\tau)r_f} \right)$$

and from that

$$\begin{aligned} V_0^l &= 249.692 \\ &+ 0.1 \times 0.5 \times \left(\frac{100}{1+0.15} - \frac{(1-0.5)\times 40}{1+(1-0.5)\times 0.1} \right) \times \left(1 + \left(\frac{1+0.1}{1+(1-0.5)\times 0.1} \right)^2 \right) \\ &+ 0.1 \times 0.5 \times \left(\frac{110}{(1+0.15)^2} - \frac{(1-0.5)\times 40}{(1+(1-0.5)\times 0.1)^2} \right) \times \left(1 + \frac{1+0.1}{1+(1-0.5)\times 0.1} \right) . \end{aligned}$$

That results in

$$V_0^l \approx 263.47$$

for the levered firm.

3.3.7 Retention Based on Market Value

Valuation Equation A retention policy could take the following form: management withholds a fixed share of the levered firm's value each year. The greater the firm's value, the greater the retained amount. The idea is to finance the firm's growth internally at a constant intensity.

Definition 3.5 (Retention-value ratio) *The firm is following a retention policy based on market values if the relation of retention and value of the firm*

$$\widetilde{l}_t = \frac{\widetilde{A}_t}{\widetilde{V}_t^l}$$

is deterministic.

We again use the symbol for leverage here, since, as we will soon see, an equation equivalent to the WACC approach can be derived with the help of this ratio. This justifies the designation of this retention policy.

Theorem 3.10 (Retention based on market values) *In the case of a retention policy based on market values, the following is valid for the levered firm that bears no debt*

$$\widetilde{V}_t^l = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\prod_{h=t+1}^{s-1}\left(1-\left(1-\tau^D\right)l_h\right)\widetilde{CF}_s^u\right]}{\prod_{h=t}^{s-1}\left(1+k_h\right)},$$

whereby

$$1+k_h = \left(1+k^{E,u}\right)\left(1-\frac{\left(1+r_f\right)\left(1-\tau^D\right)}{1+r_f\left(1-\tau^I\right)}l_h\right).$$

The last equation resembles the Miles–Ezzell formula.[20] In order to prove Theorem 3.10, we use Eq. (3.7) for $\widetilde{V}_{t+1}^l$,

$$\widetilde{V}_{t+1}^l - \widetilde{V}_{t+1}^u = \left(1-\tau^D\right)\widetilde{A}_{t+1} + \frac{\mathrm{E}_{t+1}^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \ldots + \frac{\mathrm{E}_{t+1}^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t-1}},$$

out of which with the rule for the iterated expectation

$$\frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l - \widetilde{V}_{t+1}^u\right]}{1+r_f\left(1-\tau^I\right)} = \frac{\mathrm{E}_t^Q\left[\left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \frac{\mathrm{E}_t^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^2} + \ldots + \frac{\mathrm{E}_t^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}}$$

results. When we now add the expression $\frac{\mathrm{E}_t^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_t-\left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \left(1-\tau^D\right)\widetilde{A}_t$ to both sides, then we can again under application of (3.7) bring the above equation into the form

$$\frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l - \widetilde{V}_{t+1}^u + \left(1-\tau^D\right)\left(1+r_f\right)\widetilde{A}_t - \left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} = \widetilde{V}_t^l - \widetilde{V}_t^u.$$

With the help of the Fundamental Theorem 3.2, we can eliminate $\widetilde{V}_t^u$ and $\widetilde{V}_{t+1}^u$ from the last equation for the unlevered firm and get

[20] See Theorem 2.13.

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u - \left(1-\tau^D\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \frac{\left(1-\tau^D\right)\left(1+r_f\right)\widetilde{A}_t}{1+r_f\left(1-\tau^I\right)}\ .$$

Since the retention-value ratio is deterministic, we have

$$\widetilde{V}_t^l - \frac{\left(1-\tau^D\right)\left(1+r_f\right)l_t\widetilde{V}_t^l}{1+r_f\left(1-\tau^I\right)} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u - \left(1-\tau^D\right)l_{t+1}\widetilde{V}_{t+1}^l\right]}{1+r_f\left(1-\tau^I\right)}\ .$$

We now divide by the factor $\left(1 - \frac{\left(1-\tau^D\right)\left(1+r_f\right)}{1+r_f\left(1-\tau^I\right)}l_t\right)$ and get the recursion equation

$$\widetilde{V}_t^l = \frac{\mathrm{E}_t^Q\left[\left(1-\left(1-\tau^D\right)l_{t+1}\right)\widetilde{V}_{t+1}^l + \widetilde{CF}_{t+1}^u\right]}{1+r_f\left(1-\tau^I\right)-\left(1-\tau^D\right)\left(1+r_f\right)l_t}\ .$$

When we apply this repeatedly, the following then results[21]

$$\widetilde{V}_t^l = \sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\prod_{h=t+1}^{s-1}\left(1-\left(1-\tau^D\right)l_h\right)\widetilde{CF}_s^u\right]}{\prod_{h=t}^{s-1}\left(1+r_f\left(1-\tau^I\right)-\left(1-\tau^D\right)\left(1+r_f\right)l_h\right)}\ .$$

With the help of Thm. 3.4, the following ensues

$$\widetilde{V}_t^l = \sum_{s=t+1}^{T}\frac{\mathrm{E}_t\left[\prod_{h=t+1}^{s-1}\left(1-\left(1-\tau^D\right)l_h\right)\widetilde{CF}_s^u\right]}{\prod_{h=t}^{s-1}\left((1+k_h^{E,u})(1-\frac{\left(1+r_f\right)\left(1-\tau^D\right)}{1+r_f\left(1-\tau^I\right)}l_h)\right)}\ .$$

That is the assertion of the statement.

3.3.8 Examples and Problems

3.3.8.1 The Finite Example

We assume a constant retention-value ratio of

$$l_0 = l_1 = l_2 = 10\%\ .$$

The following then results for the cost of equity

$$k_t = (1+k_t^{E,u})\left(1-\frac{\left(1+r_f\right)\left(1-\tau\right)}{1+r_f(1-\tau)}l_h\right)-1$$

[21] By definition, $\prod_{s=t+1}^{t} x_s = 1$.

$$= (1 + 0.15) \times \left(1 - \frac{(1 + 0.1) \times (1 - 0.5)}{1 + 0.1 \times (1 - 0.5)} \times 0.1\right) - 1 \approx 8.976\% \,.$$

With that we get the value of the levered firm with

$$\begin{aligned} V_0^l &= \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1 + k} + \frac{(1 - (1 - \tau)l_1)\,\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1 + k)^2} + \frac{(1 - (1 - \tau)l_1)(1 - (1 - \tau)l_2)\,\mathrm{E}\left[\widetilde{CF}_3^u\right]}{(1 + k)^3} \\ &\approx \frac{100}{1 + 0.08976} + \frac{(1 - (1 - 0.5) \times 0.1)110}{(1 + 0.08976)^2} + \frac{(1 - (1 - 0.5) \times 0.1)^2 \times 121}{(1 + 0.08976)^3} \\ &\approx 264.14 \,. \end{aligned}$$

In the infinite example, we again use

$$l_s = 10\%$$

resulting in

$$\begin{aligned} k_s &= (1 + k_s^{E,u})\left(1 - \frac{(1 + r_f)\,(1 - \tau)}{1 + r_f(1 - \tau)} l_s\right) - 1 \\ &= (1 + 0.2)\left(1 - \frac{(1 + 0.1) \times (1 - 0.5)}{1 + 0.1 \times (1 - 0.5)} 0.1\right) - 1 \approx 0.13714 \,. \end{aligned}$$

We thus get

$$\begin{aligned} V_0^l &= \sum_{t=1}^{\infty} \frac{\mathrm{E}\left[(1 - (1 - \tau)l)^{t-1}\widetilde{CF}_t^u\right]}{(1 + k)^t} = \frac{1}{1 - (1 - \tau)l} \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{\frac{1+k}{1-(1-\tau)l} - 1} = \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{k + (1 - \tau)l} \\ &= \frac{100}{0.13714 + (1 - 0.5) \times 0.1} = 534.35 \,. \end{aligned}$$

3.3.8.2 Problems

Problem 3.7 Consider the example of the unlevered firm in this section. Assume that the firm's funds are invested in risky assets. Let $A_0 = 0$. What are the highest possible retentions at times $t = 1, 2$? What is the firm's value if it institutes these retentions?

Problem 3.8 Consider the case where retention is invested in risk-free assets. Write down the valuation equation similar to (3.7). Determine the value of the levered and perpetual company if the firm follows an autonomous retention with constant A_t.

3.4 Further Reading

The literature on personal income tax is scarce. Miller and Modigliani (1961) dividends story is a predecessor of our handling of tax shield and distribution policy. Miller (1977) investigated an equilibrium model where a corporate and a personal income tax are present. The papers by Sick (1990), Taggart Jr. (1991) and Rashid and Amoaku-Adu (1995) considered personal income taxes in a valuation setup. The relation between tax and arbitrage (and in particular a Fundamental Theorem with income taxes) is also developed in a paper by Jensen (2009). Lally (2000) develops the DCF valuation implications with personal and corporate income tax where an imputation system is applicable. Fernández (2004) discusses the effect of different retention policies on the firm value.

The so-called "lock-in-effect" designates a situation where the owner of a company retains parts of the distributable cash flows due to personal income tax savings. If the tax rate differs across individuals people with high tax brackets hold assets with low dividends and people with low tax brackets hold assets with high dividend yields. This "clientele effect" is widely discussed in the literature.

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Chapter 4
Corporate and Personal Income Tax

Abstract This chapter extends the valuation framework to cases in which both corporate and personal income taxes apply.
Building on the earlier distinction between the two tax types, it analyzes their interaction under the classical system, in which dividends and interest are taxed differently.
In the final chapter of this book, we examine how to value a firm when taxes apply at both the investor and corporate levels. Attentive readers of Chapts. 2 and 3 might expect a discussion of alternative financing (Chapt. 2) and dividend (Chapt. 3) policies. We will not pursue that here, nor will we catalog the countless possible combinations. Instead, we confine ourselves to a tractable example. We conclude with brief guidance on how to proceed under other financing and dividend policies.

4.1 Assumptions

In this chapter we define an unlevered company as one that fulfills the following two conditions. First, it must be entirely self-financed; second, it must distribute all free cash flow to shareholders. If either attribute does not hold, we switch to discussing a levered company. As in Chapts. 2 and 3, we use the symbols u and l to distinguish the two cases.

In what follows we will begin by characterizing a tax system, which will be used in our example.

Corporate Income Tax The corporate income tax has the attributes mentioned in Chapt. 2: the tax is measured by the company's profit, the tax scale is linear, and the tax rate, denoted τ^C, is independent of time. Furthermore, taking out loans at time $t-1$ provides a tax advantage equal to the product of the tax scale and interest, $\tau^C \widetilde{I}_t$.

Personal Income Tax The personal income tax follows the model in Chapt. 3. On the assessment basis, we consider dividends and interest, which are taxed differently: dividends at rate τ^D and interest at rate τ^I. Such divergent treatment of income categories exists in several industrial nations. Both tax schedules are linear, and the rates are time-invariant. If the company does not distribute its entire free cash flow, retention generates a tax advantage of $\widetilde{A}_t$.

Interaction of Both Taxes As we excluded one of the two tax types in Chapts. 2 and 3, we did not have to address how the two interact. Here, that interaction must be considered. To that end, we adopt the perspective of a shareholder whose income consists solely of dividends. For dividends to be paid, the company in which the shareholder invests must first earn profits. These profits are subject to corporate tax. Consequently, any distribution to the shareholder has already been taxed once at the corporate level, and the company can distribute only what remains after corporate tax. If the shareholder must then pay personal income tax, the tax authorities effectively access the dividend a second time.

The legislature has two ways to address this situation. Either it treats the corporate income tax already paid (in whole or in part) as a first installment—an approach known as indirect relief,[1] or it provides no such allowance, which implies double taxation of dividends.[2] Occasionally, jurisdictions attempt to combine both concepts. As a rule, this is implemented by taxing dividends more lightly, especially relative to interest.[3]

In our examples, we assume the classical system with distinct tax rates for dividends and interest. Further guidance on handling alternative indirect relief policies or financial systems can be found in Cooper and Nyborg (2006), Husmann et al. (2006), and Lally (2000).

4.2 Identification and Evaluation of Tax Advantages

Gross and Free Cash Flows Our aim is to evaluate the tax advantages of the levered firm, especially relative to the unlevered firm. As in the preceding chapters, this can be done in two steps. First, we identify the tax advantages in period t. Second, we value these future tax advantages at the evaluation date. To determine their magnitude precisely, it is helpful to consult Fig. 4.1.

It shows how to get from gross cash flow to the free cash flow of the levered company. Under our assumption, the levered company's gross cash flow equals that of the unlevered company. The same holds for investment expenditures. Note that (as in Chapt. 3) $\widetilde{CF}_t^l$ in Fig. 4.1 does not represent all payments to investors, but only payments to shareholders.

Tax Shields Let us now have a look at the distributions of the levered company. We intend to compare these payments with those of an unlevered company. The unlevered company is able to pay the following amount to the shareholders,

$$\widetilde{CF}_t^u = \widetilde{GCF}_t - \widetilde{Tax}_t^{C,u} - \widetilde{Inv}_t - \widetilde{Tax}_t^{P,u}. \tag{4.1}$$

[1] In Italy, Japan, and New Zealand, for example, a form of indirect relief was in use as of 2004.

[2] In 2004, the "classical system" was adopted in Belgium, Denmark, and the USA.

[3] In 2004 adequate solutions to this problem were in existence in Germany, the Netherlands, and Greece.

Fig. 4.1 From pre-tax gross cash flows to post-tax free cash flow.

	Gross cash flow before taxes	$\widetilde{GCF}_t$
−	Corporate income taxes	$\widetilde{Tax}_t^C$
−	Investment expenses	$\widetilde{Inv}_t$
−	Interest (Creditor's taxable income)	$\widetilde{I}_t$
−	Debt repayments	$-\widetilde{D}_t + \widetilde{D}_{t-1}$
−	Retained earnings	$\widetilde{A}_t$
+	Cash flow from retained earnings	$(1 + \widetilde{r}_{t-1})\widetilde{A}_{t-1}$
−	Shareholder's personal income tax	$\widetilde{Tax}_t^P$
=	Shareholder's levered post-tax cash flow	$\widetilde{CF}_t^l$

Due to the indebtedness of the levered company the payments to the shareholders will diminish. In addition, a part of the cash flow is withheld. The following equation for the payments to the shareholders of the levered company can be derived from Fig. 4.1,

$$\widetilde{CF}_t^l = \widetilde{CF}_t^u - \widetilde{I}_t + \widetilde{D}_t - \widetilde{D}_{t-1} - \widetilde{A}_t + (1 + \widetilde{r}_{t-1})\widetilde{A}_{t-1} + \widetilde{Tax}_t^{C,u} - \widetilde{Tax}_t^{C,l} + \widetilde{Tax}_t^{P,u} - \widetilde{Tax}_t^{P,l} . \quad (4.2)$$

Here we can find two tax shields, a corporate income tax shield and a personal income tax shield. In our example we want to focus on a case in which both the indebtedness and the retention are constant over a specified period of time. Hence the following always applies,

$$\widetilde{D}_t = D, \quad \widetilde{A}_t = A . \quad (4.3)$$

Now, let us address the tax shields.

The earnings before taxes determine the corporate income tax. Consequently, analogously to Chapt. 2, the following applies

$$\begin{aligned}\widetilde{Tax}_t^{C,l} &= \tau^C \widetilde{EBT}_t^l \\ &= \tau^C \left(\widetilde{EBT}_t^u - r_f D + \widetilde{r}_{t-1} A\right) \\ &= \widetilde{Tax}_t^{C,u} - \tau^C r_f D + \tau^C \widetilde{r}_{t-1} A .\end{aligned}$$

The personal income tax follows the classical system and is based on distributions. When we look more closely at the underlying items, we see that the shareholder's tax base in the levered company decreases by the amount of interest payments and increases by the proceeds from retention. For the creditors, however, the tax advantages from borrowing must be added. These are partly risky and partly risk-free. Thus, the appropriate tax rate must be applied as follows,

$$\begin{aligned}\widetilde{Tax}_t^{P,l} &= \widetilde{Tax}_t^{P,u} - \tau^I r_f D + \tau^D \widetilde{r}_{t-1} A + \tau^I \tau^C r_f D - \tau^D \tau^C \widetilde{r}_{t-1} A \\ &= \widetilde{Tax}_t^{P,u} - \tau^I \left(1-\tau^C\right) r_f D + \tau^D \left(1-\tau^C\right) \widetilde{r}_{t-1} A \ .\end{aligned}$$

Finally, both equations add up to the entire tax shield of the levered company.[4] For this purpose we will concentrate on Eq. (4.2) and will take advantage of the fact that indebtedness and distribution remain unchanged over time,

$$\begin{aligned}\widetilde{CF}_t^l &= \widetilde{CF}_t^u - r_f D + \widetilde{r}_{t-1} A + \widetilde{Tax}^{C,u} - \widetilde{Tax}^{C,l} + \widetilde{Tax}^{P,u} - \widetilde{Tax}^{P,l} \\ &= \widetilde{CF}_t^u - \left(1-\tau^I\right)\left(1-\tau^C\right) r_f D + \left(1-\tau^D\right)\left(1-\tau^C\right) \widetilde{r}_{t-1} A \\ \mathrm{E}_{t-1}^Q\left[\widetilde{CF}_t^l\right] &= \mathrm{E}_{t-1}^Q\left[\widetilde{CF}_t^u\right] + \left(1-\tau^D\right)\left(1-\tau^C\right) r_f A \\ &\quad - \left(1-\tau^I\right)\left(1-\tau^C\right) r_f D \ .\end{aligned}$$

The difference of both equations is now easy to determine. All we need to do is to make use of the Fundamental Theorem (3.2), which applies to the levered as well as to the unlevered companies.

Once again, we must account for an important detail already mentioned in the previous chapter. We denoted the cash flow of the unlevered company—one that follows a full payout policy—by $\widetilde{CF}_t^u$. As noted earlier, this refers only to the cash flow accruing to the owners; payments to creditors (interest and/or amortization) have so far been excluded. To determine not only the value of equity but also the value of the entire firm, we must include payments to all investors. It would therefore be incorrect to focus solely on $\widetilde{CF}_t^u$.

Here we technically have two possibilities: on the one hand, we could focus on the owner's income and then add $\widetilde{D}_t$. On the other hand, we could calculate the payments to all financiers by adding the interest payments and amortization to $\widetilde{CF}_t^u$, and then subtracting the personal income tax that creditors must pay. Both methods lead to the same result, by the Fundamental Theorem of Asset Pricing. Correspondingly, this applies to the levered company as well.

According to this, the difference between the values of the companies concerned is equal to the sum of the discounted tax shields plus the market value of the debt,

$$\begin{aligned}\widetilde{V}_t^l &= \widetilde{V}_t^u + D + \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t^Q\left[\left(1-\tau^D\right)\left(1-\tau^C\right) r_f A - \left(1-\tau^I\right)\left(1-\tau^C\right) r_f D\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}} \qquad (4.4)\\ &= \widetilde{V}_t^u + D + \sum_{s=t+1}^{\infty} \frac{\left(1-\tau^D\right)\left(1-\tau^C\right)}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}} r_f A - \sum_{s=t+1}^{\infty} \frac{\left(1-\tau^I\right)\left(1-\tau^C\right)}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}} r_f D \\ &= \widetilde{V}_t^u + D + \frac{\left(1-\tau^D\right)\left(1-\tau^C\right)}{r_f\left(1-\tau^I\right)} r_f A - \frac{\left(1-\tau^I\right)\left(1-\tau^C\right)}{r_f\left(1-\tau^I\right)} r_f D \\ &= \widetilde{V}_t^u + \frac{\left(1-\tau^D\right)\left(1-\tau^C\right)}{1-\tau^I} A + \tau^C D \ .\end{aligned}$$

[4] The argumentation is equivalent to the Eq. (3.2).

This describes a more general view of the findings of the precedent chapters (Modigliani-Miller Theorem 2.7 and Thm. 3.6).

4.3 Conclusion

In the previous paragraph, we valued a company assuming that taxes are levied at both the corporate and investor levels. Under specific assumptions about the firm's lifespan and its debt and dividend policies, we derived a simple valuation equation. The remaining question is how to proceed when the debt and dividend policies deviate substantially from those assumptions.

For this purpose we analyze Eq. (4.4). Under our specific assumptions, it allows us to assess the value difference between the levered and unlevered firm. With different assumptions about debt and dividend policies, the equation can be adapted by identifying the tax advantages associated with the chosen policies. The expectations then apply to future debt $\widetilde{D}_t$ and future retention $\widetilde{A}_t$, and must be taken under the risk-neutral measure Q. In practice, however, the evaluator typically does not know this measure, leaving the equation elegant but not operational.

In a first step toward deriving a valuation equation (under the subjective probability measure), it is important to posit a linear relation between the future amounts of debt $\widetilde{D}_t$, the future amounts of retention $\widetilde{A}_t$, and the unlevered firm's cash flow $\widetilde{CF}^u_t$.[5] If we succeed in this step, the valuation equation can be written as follows,

$$V_0^l = V_0^u + x_0 D_0 + x_0' A_0 + \frac{x_1 \operatorname{E}^Q\left[\widetilde{CF}^u_1\right]}{\left(1 + r_f\left(1 - \tau^I\right)\right)^2} + \ldots + \frac{x_{T-1} \operatorname{E}^Q\left[\widetilde{CF}^u_{T-1}\right]}{\left(1 + r_f\left(1 - \tau^I\right)\right)^T} .$$

In this case, we assume that the current amount of debt and the current amount of retention are given as known factors. The parameters x_t are expressions of deterministic variables that describe the linear relation between the amounts of debt and retention, on the one hand, and the cash flow of the unlevered firm, on the other. Unfortunately, we cannot characterize them more precisely at this level of generality. With this, we reduce the evaluation of the tax shield to determining the expected values of future cash flow under Q.

Finally, we need to eliminate the risk-neutral probability measure. To do so, we use, in a second step, the assumption that the free cash flow of the unlevered company is martingale-like. Under these conditions, the cost of capital of the unlevered company provides suitable discount rates. We can then obtain realistic valuation equations that rely only on known variables (subjectively expected cash flow, cost of capital of the unlevered company, tax rates, and interest rates).

Consequently, by proceeding as described, we can value firms operating under alternative financing and dividend policies. First, we must identify the resulting tax advantages. Next, the future amounts of debt $\widetilde{D}_t$ and retention $\widetilde{A}_t$ should be linked

[5] For a similar approach in continuous time see Grinblatt and Liu (2008, Chapt. 1.C).

linearly to the unlevered firm's future cash flow. Then we evaluate the corresponding expectations of this cash flow, using the unlevered firm's cost of capital as discount rates.

But the limits of our approach are evident. Whenever we cannot construct a linear relation, the concept collapses. Such situations are easy to imagine—for example, when a firm's future investments follow a stochastic process independent of cash flow and managers commit to financing investments exclusively with equity. In that case, the future amounts of debt cannot be expressed in a linear relation with cash flows, and our approach will not yield the desired result in which the tax shield is determined solely by the cost of capital and the subjectively expected values of future cash flows.

4.4 Problem

Problem 4.1 Evaluate the value of the levered company in the infinite example using all our assumptions (from autonomous debt and retention) so far.

4.5 References

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Chapter 5
Proofs

Abstract This chapter contains proofs of the main theorems.

5.1 Williams/Gordon-Shapiro Formula (Thm. 2.2) and Equivalence of Valuation Concepts (Thm. 2.3)

Proof. We start with the proof of Thm. 2.2. From the valuation equation (Thm. 2.1) and the Assump. 2.1

$$\widetilde{V}_t^u = \sum_{s=t+1}^{T} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{\left(1+k^{E,u}\right)^{s-t}} = \sum_{s=t+1}^{T} \frac{(1+g)^{s-t}\,\widetilde{CF}_t^u}{(1+k^{E,u})^{s-t}} = \widetilde{CF}_t^u \sum_{s=t+1}^{T} \left(\frac{1+g}{1+k^{E,u}}\right)^{s-t} .$$

But this equation says exactly that the firm value is a deterministic multiple of the cash flow. Using transversality this also applies for $T \to \infty$ where the sum converges to $\frac{1+g}{k^{E,u}-g}$. Thus the Thm. 2.2 is proven. □

Now we prove Thm. 2.3. The following results from the definition of the cost of equity of the unlevered firm if $t+1 < T$

$$\begin{aligned}
\widetilde{V}_t^u &= \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u + \widetilde{V}_{t+1}^u\right]}{1+k_t^{E,u}} && \text{by Def. 2.1} \quad (5.1)\\
&= \frac{\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u + \frac{1+g}{k^{E,u}-g}\widetilde{CF}_{t+1}^u\right]}{1+k_t^{E,u}} && \text{by Thm. 2.2}\\
&= \frac{\left(1+\frac{1+g}{k^{E,u}-g}\right)\mathrm{E}_t\left[\widetilde{CF}_{t+1}^u\right]}{1+k_t^{E,u}} .
\end{aligned}$$

The following is likewise valid if $t+1 < T$

$$\widetilde{V}_t^u = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+r_f} \qquad \text{by Thm. 1.2} \tag{5.2}$$
$$= \frac{\left(1+\frac{1+g}{k^{E,u}-g}\right)\mathrm{E}_t^Q\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f} \qquad \text{by Thm. 2.2}\,.$$

The comparison of both terms results in

$$\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{1+k^{E,u}} = \frac{\mathrm{E}_t^Q\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f}$$

and this also holds for $t+1=T$ from transversality. And that is already the proposition of the theorem for $s=t+1$.

We go back to Eq.(5.1) and (5.2) and remove the terms already shown to be identical. There then remains

$$\frac{\mathrm{E}_t\left[\widetilde{V}^u_{t+1}\right]}{1+k^{E,u}} = \frac{\mathrm{E}_t^Q\left[\widetilde{V}^u_{t+1}\right]}{1+r_f}$$

or

$$\frac{\mathrm{E}_t\left[\frac{\mathrm{E}_{t+1}\left[\widetilde{CF}^u_{t+2}+\widetilde{V}^u_{t+2}\right]}{1+k^{E,u}_{t+1}}\right]}{1+k^{E,u}} = \frac{\mathrm{E}_t^Q\left[\frac{\mathrm{E}^Q_{t+1}\left[\widetilde{CF}^u_{t+2}+\widetilde{V}^u_{t+2}\right]}{1+r_f}\right]}{1+r_f}\,.$$

The law of the iterated expectation as well as the fact that the dividend-price relation is deterministic establishes

$$\frac{\mathrm{E}_t\left[\left(1+\frac{1+g}{k^{E,u}-g}\right)\widetilde{CF}^u_{t+2}\right]}{(1+k^{E,u})^2} = \frac{\mathrm{E}_t^Q\left[\left(1+\frac{1+g}{k^{E,u}-g}\right)\widetilde{CF}^u_{t+2}\right]}{(1+r_f)^2}\,.$$

After shortening of $\left(1+\frac{1+g}{k^{E,u}-g}\right)$, that is the claim of the theorem for $s=t+2$. The propositions for $s=t+3,\ldots$ can now be proven analogously. □

5.2 Valuation Formula with Investment Policy Based on Cash Flows (Thm. 2.19)

Proof. With the following proof you have to make an effort to keep an overview. We begin with showing the difference between investments and accruals. Because of Def. 2.10 and because there are only non discretionary accruals, for the time being we could write

$$\widetilde{Inv}_t - \widetilde{Accr}_t = \widetilde{Inv}_t - \frac{1}{n}\left(\widetilde{Inv}_{t-1} + \ldots + \widetilde{Inv}_{t-n}\right)\,.$$

Now it makes sense to use Def. 2.13 and replace $\widetilde{Inv}_s$ with $\alpha_s \widetilde{CF}^u_s$. That, however, fails in that for investment amounts that are not in the future, we have to take historical real numbers and can only use free cash flows in relation to future investments. With

$$\widetilde{H}_s = \begin{cases} \alpha_s \widetilde{CF}^u_s, & \text{if } s > 0 \\ Inv_s, & \text{else} \end{cases}$$

we get for all $t \geq 1$ equation

$$\widetilde{Inv}_t - \widetilde{Accr}_t = \widetilde{H}_t - \frac{1}{n}\left(\widetilde{H}_{t-1} + \ldots + \widetilde{H}_{t-n}\right) \ . \tag{5.3}$$

Taking advantage of this relation, we get the following for the firm's book value using Thm. 2.15 (operating assets relation) and Assump. 2.9 (subscribed capital) as well as Eq. (2.35)

$$\begin{aligned}
\underline{\widetilde{V}}^l_t &= \underline{V}^l_0 + \underline{e}^l_{0,1} + \ldots + \underline{e}^l_{t-1,t} + \left(\widetilde{Inv}_1 - \widetilde{Accr}_1\right) + \ldots + \left(\widetilde{Inv}_t - \widetilde{Accr}_t\right) \\
&= \underline{V}^l_0 + \underline{e}^l_{0,t} + \left(\widetilde{Inv}_1 - \widetilde{Accr}_1\right) + \ldots + \left(\widetilde{Inv}_t - \widetilde{Accr}_t\right) \\
&= \underline{V}^l_0 + \underline{e}^l_{0,t} + \sum_{s=1}^{t}\left(\widetilde{H}_s - \frac{1}{n}\sum_{r=s-1}^{s-n}\widetilde{H}_r\right) \\
&= \underline{V}^l_0 + \underline{e}^l_{0,t} + \sum_{s=1}^{t}\widetilde{H}_s - \frac{1}{n}\sum_{s=1}^{t}\sum_{r=s-n}^{s-1}\widetilde{H}_r \\
&= \underline{V}_0 + \underline{e}^l_{0,t} + \sum_{s=1}^{t}\widetilde{H}_s - \frac{1}{n}\sum_{s=1}^{t}\sum_{r=0}^{n-1}\widetilde{H}_{r+s-n} \ .
\end{aligned} \tag{5.4}$$

We will now rearrange the double sum. To this end we determine the number of possibilities to represent a given number a as a sum $a = r + s$ such that the first summand r is between 0 and $n - 1$ and the second summand s is between 1 and t. We first show that $A(a)$ is given by

$$A(a) := \begin{cases} a & \text{if} \quad 0 < a < \min(t, n), \\ \min(t, n) & \text{if} \quad \min(t, n) \leq a \leq \max(t, n), \\ n + t - a & \text{if} \quad \max(t, n) < a < n + t, \\ 0 & \text{else.} \end{cases}$$

To this end write $a = r + s$ as a sum of ones with a separating vertical line between r and s

$$\underbrace{\overbrace{1 \quad 1 \quad 1 \quad 1}^{r} \,|\, \overbrace{1 \quad 1 \quad 1 \quad 1 \quad 1}^{s}}_{a} \ .$$

The separating line cannot lie left from the first one (because $r \geq 0$) and cannot lie right from the last one (because $s > 0$). Hence, there are exactly $A(a) = a$ possibilities; this explains the first row of our definition. If a increases by one the quantity $A(a)$ increases by one as well.

If a gets above $\min(n, t)$ then $A(a)$ remains at its current level. This is so because the vertical line cannot occupy all available positions. If, for example $n \leq t$ and therefore $n < a$, the vertical line cannot be right from the n-th one since we must have $r < n$. The argument is analog for $n > t$. This explains the second line of the definition.

The third line of the definition can be understood as follows. Look at a representation $r + s$ where the vertical line is farthest to the left. Because a is greater than n and t by adding an additional one to the right this representation violates our rules since s gets too large. Hence, increasing a by one decreases the number $A(a)$ by one. This finishes our proof of $A(a)$.

Now (5.4) can be simplified to

$$\begin{aligned}
\underline{\widetilde{V}}_t &= \underline{V}_0 + \underline{e}^l_{0,t} + \sum_{a=1}^{t} \widetilde{H}_a - \frac{1}{n} \sum_{a=1}^{n+t-1} A(a) \widetilde{H}_{a-n} \\
&= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{a=1-n}^{0} \frac{A(a+n)}{n} \widetilde{H}_a + \sum_{a=1}^{t} \left(1 - \frac{A(a+n)}{n}\right) \widetilde{H}_a \\
&= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{0} \frac{A(s+n)}{n} Inv_s + \sum_{s=1}^{t} \left(1 - \frac{A(s+n)}{n}\right) \alpha_s \widetilde{CF}_s \,. \qquad (5.5)
\end{aligned}$$

This equation can be simplified even further. To this end we will look at the distinct cases.

First let $n < t$. Then from the definition of $A(s+n)$

$$\begin{aligned}
A(s+n) &= \begin{cases} s+n & \text{if} \quad 0 < s+n < n \\ n & \text{if} \quad n \leq s+n \leq t \\ n+t-s-n & \text{if} \quad t < s+n < n+t \end{cases} \\
&= \begin{cases} s+n & \text{if} \quad -n < s < 0, \\ n & \text{if} \quad 0 \leq s \leq t-n, \\ t-s & \text{if} \quad t-n < s < t. \end{cases}
\end{aligned}$$

We will treat the second and the third summand separately. Because the index s in the first summand runs from $1-n$ to 0 and the index in the second summand runs from 1 to $t-1$ (5.5) simplifies to

$$\begin{aligned}
\underline{\widetilde{V}}_t = \underline{V}_0 + \underline{e}^l_{0,t} &- \sum_{s=1-n}^{0} \frac{A(s+n)}{n} Inv_s + \sum_{s=1}^{t-n} \left(1 - \frac{A(s+n)}{n}\right) \alpha_s \widetilde{CF}_s \\
&+ \sum_{s=t-n+1}^{t} \left(1 - \frac{A(s+n)}{n}\right) \alpha_s \widetilde{CF}_s
\end{aligned}$$

$$
\begin{aligned}
&= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{0} \frac{s+n}{n}\, Inv_s + \sum_{s=1}^{t-n} \left(1 - \frac{n}{n}\right) \alpha_s \widetilde{CF}_s \\
&\quad + \sum_{s=t-n+1}^{t} \left(1 - \frac{t-s}{n}\right) \alpha_s \widetilde{CF}_s \\
&= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{0} \frac{s+n}{n}\, Inv_s + \sum_{s=t-n+1}^{t} \frac{n-t+s}{n}\, \alpha_s \widetilde{CF}_s\ . \qquad (5.6)
\end{aligned}
$$

Let now $n \geq t$. Then from the definition of $A(s+n)$

$$
\begin{aligned}
A(s+n) &= \begin{cases} s+n & \text{if} \quad 0 < s+n < t \\ t & \text{if} \quad t \leq s+n \leq n \\ n+t-s-n & \text{if} \quad n < s+n < n+t \end{cases} \\
&= \begin{cases} s+n & \text{if} \quad -n < s < t-n \\ t & \text{if} \quad t-n \leq s \leq 0 \\ t-s & \text{if} \quad 0 < s < t\ . \end{cases}
\end{aligned}
$$

Again we will treat the second and the third summand separately. The index s in the first summand runs from $1-n$ to 0 and in the second summand from 1 to $t-1$. Hence, (5.5) simplifies to

$$
\begin{aligned}
\widetilde{\underline{V}}_t &= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{t-n} \frac{A(s+n)}{n}\, Inv_s - \sum_{s=t-n+1}^{0} \frac{A(s+n)}{n}\, Inv_s \\
&+ \quad \sum_{s=1}^{t} \left(1 - \frac{A(s+n)}{n}\right) \alpha_s \widetilde{CF}_s \\
&= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{t-n} \frac{s+n}{n}\, Inv_s - \sum_{s=t-n+1}^{0} \frac{t}{n}\, Inv_s + \sum_{s=1}^{t} \left(1 - \frac{t-s}{n}\right) \alpha_s \widetilde{CF}_s \\
&= \underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{0} \frac{\min(s+n,t)}{n}\, Inv_s + \sum_{s=1}^{t} \frac{n-t+s}{n}\, \alpha_s \widetilde{CF}_s\ . \qquad (5.7)
\end{aligned}
$$

Now we are able to rejoin both cases $n < t$ and $n \geq t$. The Equations (5.6) and (5.7) yield in compact notation

$$
\widetilde{\underline{V}}_t = \underbrace{\underline{V}_0 + \underline{e}^l_{0,t} - \sum_{s=1-n}^{0} \frac{\min(s+n,t)}{n} Inv_s}_{:=\underline{V}^{*l}_0} + \sum_{s=1+\max(t-n,0)}^{t} \frac{n-t+s}{n}\, \alpha_s \widetilde{CF}_s\ .
$$

The first three summands will be designated as $\underline{V}^{*l}_0$. Economically, this term concerns the amount which the firm's book value would be if up to time t, there are exclusively increases in subscribed capital and no single investment. That leads us to the representation

$$\underline{\widetilde{V}}_t^l = \underline{V}_0^{*l} + \sum_{s=1+\max(t-n,0)}^{t} \frac{n-(t-s)}{n}\,\alpha_s\,\widetilde{CF}_s^u\,.$$

When we use the agreement $\alpha_s = 0$ for $s \leq 0$ (α was up to now only defined for future times) this equation becomes

$$\underline{\widetilde{V}}_t^l = \underline{V}_0^{*l} + \frac{n}{n}\widetilde{CF}_t^u\,\alpha_t + \frac{n-1}{n}\widetilde{CF}_{t-1}^u\,\alpha_{t-1} + \ldots + \frac{1}{n}\widetilde{CF}_{1+t-n}^u\,\alpha_{1+t-n} \tag{5.8}$$

and we will from now on use this form.

The valuation with a policy based on book values is now finally successful by means of this representation. For that purpose we use everything that we have. Look at the valuation Eq. (2.14) which is valid for every conceivable financing policy and onto which we want to fall back now. With means of financing based on book values, we have at all times

$$\underline{\widetilde{D}}_t = \underline{l}_t\underline{\widetilde{V}}_t^l\,,$$

which with (5.8) and using Assump. 2.7 leads us to

$$\begin{aligned}
V_0^l = V_0^u &+ \tau r_f\frac{\underline{l}_0\underline{V}_0^l}{1+r_f} + \tau r_f\sum_{t=1}^{T-1}\frac{\underline{l}_t}{(1+r_f)^{t+1}}\operatorname{E}^Q\Bigg[\underline{V}_0^{*l} + \frac{n}{n}\widetilde{CF}_t^u\alpha_t \\
&+\frac{n-1}{n}\widetilde{CF}_{t-1}^u\alpha_{t-1} + \ldots + \frac{1}{n}\widetilde{CF}_{t-n+1}^u\alpha_{t-n+1}\Bigg] \\
&= \widetilde{V}_0^u + \tau r_f\frac{\underline{l}_0\underline{V}_0^l}{1+r_f} + \tau r_f\sum_{t=1}^{T-1}\underline{l}_t\left(\frac{\underline{V}_0^{*l}}{(1+r_f)^{t+1}} + \frac{\operatorname{E}^Q\left[\widetilde{CF}_t^u\right]}{(1+r_f)^t}\,\frac{\frac{n}{n}\alpha_t}{1+r_f}\right. \\
&\left.+\frac{\operatorname{E}^Q\left[\widetilde{CF}_{t-1}^u\right]}{(1+r_f)^{t-1}}\,\frac{\frac{n-1}{n}\alpha_{t-1}}{(1+r_f)^2} + \ldots + \frac{\operatorname{E}^Q\left[\widetilde{CF}_{1+t-n}^u\right]}{(1+r_f)^{1+t-n}}\,\frac{\frac{1}{n}\alpha_{1+t-n}}{(1+r_f)^n}\right).
\end{aligned}$$

We are now using the Assump. 2.1 and the Theorem 2.3 based on it. That allows for the representation

$$\begin{aligned}
V_0^l = V_0^u &+ \tau r_f\frac{\underline{l}_0\underline{V}_0^l}{1+r_f} + \tau r_f\sum_{t=1}^{T-1}\underline{l}_t\left(\frac{\underline{V}_0^{*l}}{(1+r_f)^{t+1}} + \frac{\operatorname{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t}\,\frac{\frac{n}{n}\alpha_t}{1+r_f}\right. \\
&\left.+\frac{\operatorname{E}\left[\widetilde{CF}_{t-1}^u\right]}{(1+k^{E,u})^{t-1}}\,\frac{\frac{n-1}{n}\alpha_{t-1}}{(1+r_f)^2} + \ldots + \frac{\operatorname{E}\left[\widetilde{CF}_{1+t-n}^u\right]}{(1+k^{E,u})^{1+t-n}}\,\frac{\frac{1}{n}\alpha_{1+t-n}}{(1+r_f)^n}\right) \\
&= V_0^u + \tau r_f\frac{\underline{l}_0\underline{V}_0^l}{1+r_f} + \tau r_f\sum_{t=1}^{T-1}\underline{l}_t\,\frac{\underline{V}_0^{*l}}{(1+r_f)^{t+1}}
\end{aligned}$$

$$+\tau r_f \sum_{t=1}^{T-1}\left(\frac{\alpha_t\,\mathrm{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t}\,\frac{\frac{n}{n}\underline{l}_t}{1+r_f}+\frac{\alpha_{t-1}\,\mathrm{E}\left[\widetilde{CF}_{t-1}^u\right]}{(1+k^{E,u})^{t-1}}\,\frac{\frac{n-1}{n}\underline{l}_t}{(1+r_f)^2}\right.$$

$$\left.+\ldots+\frac{\alpha_{1+t-n}\,\mathrm{E}\left[\widetilde{CF}_{1+t-n}^u\right]}{(1+k^{E,u})^{1+t-n}}\,\frac{\frac{1}{n}\underline{l}_t}{(1+r_f)^n}\right).$$

Lastly, we only have the terms in the last two lines to concentrate on. We are obviously looking at a double sum. To simplify its representation, we have to consider, how often an expected cash flow comes up. It is recognized that the coefficient in front of $\frac{\alpha_t\,\mathrm{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t}$ appears with the expressions

$$\frac{\frac{n}{n}\underline{l}_t}{1+r_f},\ \frac{\frac{n-1}{n}\underline{l}_{t+1}}{(1+r_f)^2},\ \ldots,\ \frac{\frac{1}{n}\underline{l}_{n+t-1}}{(1+r_f)^n}\ .$$

Again, this requires that coefficients $\underline{l}_s$ with an index greater than $T-1$ are set to zero. We get

$$V_0^l = V_0^u + \tau r_f\frac{\underline{l}_0\underline{V}_0^l}{1+r_f} + \tau r_f\sum_{t=1}^{T-1}\underline{l}_t\,\frac{\underline{V}_0^{*l}}{(1+r_f)^{t+1}}$$

$$++\tau r_f\sum_{t=1}^{T-1}\frac{\alpha_t\,\mathrm{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t}\left(\frac{\frac{n}{n}\underline{l}_t}{1+r_f}+\frac{\frac{n-1}{n}\underline{l}_{t+1}}{(1+r_f)^2}+\ldots+\frac{\frac{1}{n}\underline{l}_{n+t-1}}{(1+r_f)^n}\right),$$

where $\underline{l}_s = 0$ for $s \geq T$. With that Thm. 2.19 is finally proven. □

5.3 Adjusted Modigliani-Miller Formula (Thm. 2.20)

Proof. Many factors remain constant in the theorem. More than anything, that affects the debt ratio $\underline{l}$, the investment parameter α and the subscribed capital. Beyond that, it is assumed that the firm to be valued exists without end. By disregarding the time indices with the investment parameter and the debt ratio, we get the following using the sum of a geometric sequence

$$V_0^l = V_0^u + \tau r_f \frac{\underline{l} V_0^l}{1+r_f} + \tau r_f \sum_{t=1}^{\infty} \frac{\underline{l} V_0^{*l}}{(1+r_f)^{t+1}} + \tau r_f \sum_{t=1}^{\infty} \frac{\alpha \operatorname{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t} \left(\frac{\frac{n}{n}\underline{l}}{1+r_f} + \frac{\frac{n-1}{n}\underline{l}}{(1+r_f)^2} + \ldots + \frac{\frac{1}{n}\underline{l}}{(1+r_f)^n} \right)$$

$$= V_0^u + \frac{\tau \underline{l}\left(r_f \underline{V}_0^l + \underline{V}_0^{*l}\right)}{1+r_f} + \frac{\tau r_f \alpha \underline{l}}{n} \sum_{t=1}^{\infty} \frac{\operatorname{E}\left[\widetilde{CF}_t^u\right]}{\left(1+k^{E,u}\right)^t} \left(\frac{n}{1+r_f} + \frac{n-1}{(1+r_f)^2} + \ldots + \frac{1}{(1+r_f)^n} \right) . \tag{5.9}$$

We make the effort now to get a compact representation of the expression

$$\frac{n}{1+r_f} + \frac{n-1}{(1+r_f)^2} + \ldots + \frac{1}{(1+r_f)^n}.$$

To do so we look at the identity

$$(1+r_f)^n + (1+r_f)^{n-1} + \ldots + (1+r_f) = \frac{(1+r_f)((1+r_f)^n - 1)}{r_f}$$

and derive it according to r_f,

$$n(1+r_f)^{n-1} + (n-1)(1+r_f)^{n-2} + \ldots + 1 = \frac{1 + (nr_f - 1)(1+r_f)^n}{r_f^2} .$$

Multiplying by $(1+r_f)^{-n}$ results in

$$n(1+r_f)^{-1} + (n-1)(1+r_f)^{-2} + \ldots + (1+r_f)^{-n} = \frac{nr_f - 1 + (1+r_f)^{-n}}{r_f^2}$$

Entering this into Eq. (5.9), results in

$$V_0^l = V_0^u + \frac{\tau \underline{l}\left(r_f \underline{V}_0^l + \underline{V}_0^{*l}\right)}{1+r_f} + \frac{\tau r_f \alpha \underline{l}}{n} \frac{nr_f - 1 + (1+r_f)^{-n}}{r_f^2} \sum_{t=1}^{\infty} \frac{\operatorname{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t} .$$

We recognize that the sum on the right hand side exactly corresponds to the market value of the unlevered firm. This leads us to

$$V_0^l = V_0^u + \frac{\tau \underline{l}\left(r_f \underline{V}_0^l + \underline{V}_0^{*l}\right)}{1+r_f} + \frac{nr_f - 1 + (1+r_f)^{-n}}{nr_f} \tau \alpha \underline{l} V_0^u .$$

Lastly we turn our attention to the term

$$\frac{\tau \underline{l} \left(r_f \underline{V}_0^l + \underline{V}_0^{*l} \right)}{1 + r_f}$$

and consider that at time $t = 0$ the identity

$$\underline{V}_0^{*l} = \underline{V}_0^l - \sum_{s=1-n}^{0} \frac{\min(n-s,t)}{n} Inv_s$$

applies, so long as the influence of depreciation on those assets raised before time $t = 0$ is not excluded. But when we disregard this influence according to the gotten condition, then we get

$$\frac{\tau \underline{l} \left(r_f \underline{V}_0^l + \underline{V}_0^{*l} \right)}{1 + r_f} = \frac{\tau \underline{l} \left(r_f \underline{V}_0^l + \underline{V}_0^l \right)}{1 + r_f} = \tau \underline{l} \underline{V}_0^l \ .$$

We recognize that the product of debt ratio and book value of the value of the firm corresponds to the book value of the debt, and can close up the proof considering Assump. 2.7. □

5.4 Valuation Formula with Financing Based on Cash Flows (Thm. 2.21 and 2.22)

Proof. From the definition of financing based on cash flows, the following first results for the amount of debt using the fact that debt is ris-free

$$\widetilde{D}_t = \left((1 + \alpha r_f (1-\tau)) D_0 - \alpha \, \widetilde{CF}_1^u \right)^+ \qquad \forall t \geq 1 \ .$$

We enter this into Eq. (2.14) and get

$$\begin{aligned} V_0^l &= V_0^u + \frac{\tau \, \mathrm{E}^Q[Z_1]}{1 + r_f} + \sum_{t=1}^{T-1} \frac{\tau r_f \, \mathrm{E}^Q \left[\left((1 + \alpha r_f (1-\tau)) D_0 - \alpha \, \widetilde{CF}_1^u \right)^+ \right]}{(1 + r_f)^{t+1}} \\ &= V_0^u + \frac{\tau r_f D_0}{1 + r_f} + \frac{\tau r_f \, \mathrm{E}^Q \left[\left((1 + \alpha r_f (1-\tau)) D_0 - \alpha \, \widetilde{CF}_1^u \right)^+ \right]}{(1 + r_f) r_f} \left(1 - \frac{1}{(1 + r_f)^{T-1}} \right) . \end{aligned}$$

With help of Thm. 2.2, the third summand can be further simplified

$$V_0^l = V_0^u + \frac{\tau r_f D_0}{1 + r_f} + \frac{(1+g)\tau\alpha}{k^{E,u} - g} \frac{\mathrm{E}^Q \left[\left(\frac{1 + \alpha r_f (1-\tau)}{\alpha \frac{1+g}{k^{E,u} - g}} D_0 - \widetilde{V}_1^u \right)^+ \right]}{1 + r_f} \left(1 - \frac{1}{(1 + r_f)^{T-1}} \right) .$$

Let us now look at a put on the value of the unlevered firm at time $t = 1$ with a strike of $\frac{1 + \alpha r_f (1-\tau)}{\alpha \frac{1+g}{k^{E,u} - g}} D_0$. The bearer of this option receives the difference of the exercise price and the firm value, if this difference is positive. In the opposite case, the payment comes to zero. To determine the value of this put Π, we have to evaluate the expectation $E^Q[\cdot]$ of the payments of the put and discount them at the risk-free rate according to the duration of the option. This results exactly in

$$\Pi = \frac{\mathrm{E}^Q\left[\left(\frac{1+\alpha r_f(1-\tau)}{\alpha\frac{1+g}{k^{E,u}-g}} D_0 - \widetilde{V}_1^u\right)^+\right]}{1+r_f} .$$

But with that applies

$$V_0^l = V_0^u + \frac{\tau r_f D_0}{1+r_f} + \tau\alpha\frac{1+g}{k^{E,u}-g}\Pi\left(1-\frac{1}{(1+r_f)^{T-1}}\right) ,$$

and that was what we wanted to show.

We come to the proof of Thm. 2.22. Since the amount of debt of the first period is positive, expression $(1+\alpha r_f(1-\tau))D_0 - \alpha\,\widetilde{CF}_1^u$ will not be negative in any case. Therefore we can write

$$\widetilde{D}_t = (1+\alpha r_f(1-\tau))D_0 - \alpha\,\widetilde{CF}_1^u$$

for all $t \geq 1$. From that we get the simpler valuation equation

$$\begin{aligned} V_0^l &= V_0^u + \frac{\tau r_f D_0}{1+r_f} + \frac{\tau r_f\,\mathrm{E}^Q\left[(1+\alpha r_f(1-\tau))D_0 - \alpha\,\widetilde{CF}_1^u\right]}{(1+r_f)r_f}\left(1-\frac{1}{(1+r_f)^{T-1}}\right) \\ &= V_0^u + \frac{\tau r_f D_0}{1+r_f} + \left(\frac{\tau(1+\alpha r_f(1-\tau))D_0}{1+r_f} - \tau\alpha\frac{\mathrm{E}^Q\left[\widetilde{CF}_1^u\right]}{1+r_f}\right)\left(1-\frac{1}{(1+r_f)^{T-1}}\right) . \end{aligned}$$

Due to the equivalence of the valuation concepts (Thm. 2.3), there results from that

$$V_0^l = V_0^u + \frac{\tau r_f D_0}{1+r_f} + \left(\frac{\tau(1+\alpha r_f(1-\tau))D_0}{1+r_f} - \tau\alpha\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}}\right)\left(1-\frac{1}{(1+r_f)^{T-1}}\right) .$$

Lastly we make use of the fact that expected cash flows are constant. With that we end up with

$$\begin{aligned} V_0^l = V_0^u + \frac{\tau r_f D_0}{1+r_f} &+ \frac{\tau(1+\alpha r_f(1-\tau))D_0}{1+r_f}\left(1-\frac{1}{(1+r_f)^{T-1}}\right) \\ &- \tau\alpha\frac{V_0^u k^{E,u}}{1+k^{E,u}}\frac{1-\frac{1}{(1+r_f)^{T-1}}}{1-\frac{1}{(1+k^{E,u})^{T-1}}} . \end{aligned}$$

That agrees with the claim. □

5.5 Valuation with Financing Based on Dividends (Thm. 2.23)

Proof. Here we apply a different method to establish the value of the firm. For that we concentrate on the payments, which go to the debt and equity financiers. From the Fundamental Theorem the following first results for the levered firm

$$\begin{aligned} V_0^l &= \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{CF}_t^u + \tau \widetilde{I}_t\right]}{(1+r_f)^t} \\ &= \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{CF}_t^u - \widetilde{D}_{t-1} - (1-\tau)\widetilde{I}_t + \widetilde{D}_t\right]}{(1+r_f)^t} + \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{I}_t + \widetilde{D}_{t-1} - \widetilde{D}_t\right]}{(1+r_f)^t} . \end{aligned}$$

The second summand agrees with the sum of the discounted payments to the debt financiers. It should be exactly equal to D_0, which can be easily proven,

$$\begin{aligned} \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{I}_t + \widetilde{D}_{t-1} - \widetilde{D}_t\right]}{(1+r_f)^t} &= \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{I}_t + \widetilde{D}_{t-1}\right]}{(1+r_f)^t} - \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{D}_t\right]}{(1+r_f)^t} \\ &= \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{D}_{t-1}\right]}{(1+r_f)^{t-1}} - \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{D}_t\right]}{(1+r_f)^t} \\ &= D_0 . \end{aligned}$$

With that we have the following equation for the value of the levered firm,

$$V_0^l = \sum_{t=1}^{T} \frac{\mathrm{E}^Q\left[\widetilde{CF}_t^u - \widetilde{D}_{t-1} - (1-\tau)\widetilde{I}_t + \widetilde{D}_t\right]}{(1+r_f)^t} + D_0 .$$

During the first n periods, the shareholders get exactly the amount Div. Afterwards, the amount of debt remains constant. Using (2.20) that leads to

$$\begin{aligned} V_0^l &= D_0 + \sum_{t=0}^{n-1} \frac{Div}{(1+r_f)^{t+1}} + \sum_{t=n}^{T-1} \frac{\mathrm{E}^Q\left[\widetilde{CF}_{t+1}^u - (1-\tau)\widetilde{I}_{n+1}\right]}{(1+r_f)^{t+1}} - \frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^T} \\ &= D_0 + \left(1 - \frac{1}{(1+r_f)^n}\right)\frac{Div}{r_f} + \sum_{t=n}^{T-1} \frac{\mathrm{E}^Q\left[\widetilde{CF}_{t+1}^u\right]}{(1+r_f)^{t+1}} \\ &\qquad - \frac{(1-\tau)r_f\,\mathrm{E}^Q\left[\widetilde{D}_n\right]}{r_f(1+r_f)^n}\left(1 - \frac{1}{(1+r_f)^{T-n}}\right) - \frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^T} . \end{aligned}$$

When we make use of the equivalence of the valuation concepts from (Theorem 2.3) and further consider that the rate of growth of the expected cash flows is constant ($g =$ const.), there then results

$$V_0^l = D_0 + \left(1 - \frac{1}{(1+r_f)^n}\right)\frac{Div}{r_f} + \sum_{t=n}^{T-1}\frac{(1+g)^t\,\mathrm{E}\left[\widetilde{CF}_1^u\right]}{(1+k^{E,u})^{t+1}} - \frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^n}\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right) \tag{5.10}$$

$$= D_0 + \left(1 - \frac{1}{(1+r_f)^n}\right)\frac{Div}{r_f} + \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{k^{E,u}-g}\left[\left(\frac{1+g}{1+k^{E,u}}\right)^n - \left(\frac{1+g}{1+k^{E,u}}\right)^T\right] - \frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^n}\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right). \tag{5.11}$$

We now turn to the amount of debt at time n. Since according to the condition that the credit always remains positive, we can simplify the formation law of the debt. With (2.20) and $r_f^* = r_f(1-\tau)$ applies

$$\mathrm{E}_{t-1}^Q\left[\widetilde{D}_t\right] = \mathrm{E}_{t-1}^Q\left[Div - \widetilde{CF}_t^u\right] + (1+r_f^*)D_{t-1} \qquad \forall t \le n\,.$$

Taking advantage of the recursion relationship leads to

$$\mathrm{E}^Q\left[\widetilde{D}_n\right] = \mathrm{E}^Q\left[Div - \widetilde{CF}_n^u\right] + (1+r_f^*)\ \mathrm{E}^Q\left[Div - \widetilde{CF}_{n-1}^u\right] + \ldots + (1+r_f^*)^{n-1}\ \mathrm{E}^Q\left[Div - \widetilde{CF}_1^u\right] + (1+r_f^*)^n D_0\,.$$

This equation can be brought into the form

$$\mathrm{E}^Q\left[\widetilde{D}_n\right] = \frac{(1+r_f^*)^n - 1}{r_f^*}Div + (1+r_f^*)^n D_0 - \mathrm{E}^Q\left[\widetilde{CF}_n^u\right] - (1+r_f^*)\ \mathrm{E}^Q\left[\widetilde{CF}_{n-1}^u\right] - \ldots - (1+r_f^*)^{n-1}\ \mathrm{E}^Q\left[\widetilde{CF}_1^u\right].$$

From that the following is valid under the discounted expectation

$$\frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^n} = \frac{(1+r_f^*)^n - 1}{r_f^*(1+r_f)^n}Div + \left(\frac{1+r_f^*}{1+r_f}\right)^n D_0 - \frac{\mathrm{E}^Q\left[\widetilde{CF}_n^u\right]}{(1+r_f)^n} - \frac{1+r_f^*}{1+r_f}\frac{\mathrm{E}^Q\left[\widetilde{CF}_{n-1}^u\right]}{(1+r_f)^{n-1}} - \ldots - \frac{(1+r_f^*)^{n-1}}{(1+r_f)^{n-1}}\frac{\mathrm{E}^Q\left[\widetilde{CF}_1^u\right]}{1+r_f}$$

or due to Thm. 2.3 as well as the constant rate of growth g and with $\gamma = \frac{1+r_f^*}{1+r_f}$ and $\delta = \frac{1+g}{1+k^{E,u}}$

$$\frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^n} = \frac{(1+r_f^*)^n-1}{r_f^*(1+r_f)^n}Div + \gamma^n D_0 \\ - \delta^n \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+g} - \gamma\delta^{n-1}\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+g} - \ldots - \gamma^{n-1}\delta\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+g} .$$

We combine the last summands in the equation and get

$$\frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^n} = \frac{(1+r_f^*)^n-1}{r_f^*(1+r_f)^n}Div + \gamma^n D_0 - \left(1+\frac{\gamma}{\delta}+\ldots+\left(\frac{\gamma}{\delta}\right)^{n-1}\right)\delta^n\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+g} .$$

or, after simplification,

$$\frac{\mathrm{E}^Q\left[\widetilde{D}_n\right]}{(1+r_f)^n} = \frac{(1+r_f^*)^n-1}{r_f^*(1+r_f)^n}Div + \gamma^n D_0 - \frac{\gamma^n-\delta^n}{\frac{\gamma}{\delta}-1}\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+g} .$$

We put this into (5.11) and get

$$V_0^l = \left(1-\gamma^n\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right)\right)D_0 \\ + \left(1-\gamma^n\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right)-\tau\left(1-\frac{1}{(1+r_f)^T}\right)\right)\frac{Div}{r_f(1-\tau)} \\ + \left(\delta^n-\delta^T+\frac{\gamma^n-\delta^n}{\frac{\gamma}{\delta}-1}\frac{k^{E,u}-g}{1+g}\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right)\right)\frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{k^{E,u}-g} . \quad (5.12)$$

We now take advantage of the expected cash flow of the unlevered firm showing a constant rate of growth. It follows that

$$V_0^u = \sum_{t=1}^T \frac{\mathrm{E}\left[\widetilde{CF}_t^u\right]}{(1+k^{E,u})^t} = \sum_{t=1}^T \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+g}\left(\frac{1+g}{1+k^{E,u}}\right)^t = \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{k^{E,u}-g}\left(1-\left(\frac{1+g}{1+k^{E,u}}\right)^T\right) .$$

Plugging this into (5.12) gives

$$V_0^l = \left(1-\gamma^n\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right)\right)D_0 \\ + \left(1-\gamma^n\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right)-\tau\left(1-\frac{1}{(1+r_f)^T}\right)\right)\frac{Div}{r_f(1-\tau)} \\ + \left(\delta^n-\delta^T+\frac{\gamma^n-\delta^n}{\frac{\gamma}{\delta}-1}\frac{k^{E,u}-g}{1+g}\left(1-\tau\left(1-\frac{1}{(1+r_f)^{T-n}}\right)\right)\right)\frac{V_0^u}{1-\delta^T} .$$

When we sort the terms around a bit, then we get our desired result. □

5.6 Valuation with Debt-Cash Flow Ratio (Thm. 2.24 and Thm. 2.25)

Proof. From the definition of the dynamic leverage ratio we get

$$\widetilde{D}_s = \widetilde{L}^d_s \left(\widetilde{CF}^u_s + \tau r_f \widetilde{D}_{s-1} \right) .$$

and hence

$$\widetilde{D}_s = \widetilde{L}^d_s \left(\widetilde{CF}^u_s + \tau r_f \widetilde{L}^d_{s-1} \left(\widetilde{CF}^u_{s-1} + \tau r_f \widetilde{D}_{s-2} \right) \right) .$$

Using induction this gives for $s > t$

$$\widetilde{D}_s = (\tau r_f)^{s-t} \widetilde{L}^d_s \dots \widetilde{L}^d_{t+1} \widetilde{D}_t + \sum_{u=t+1}^{s} \widetilde{L}^d_s \dots \widetilde{L}^d_u (\tau r_f)^{s-u} \, \widetilde{CF}^u_u .$$

Plugging this into the general valuation formula (2.14) we get

$$\widetilde{V}^l_t = \widetilde{V}^u_t + \frac{\tau r_f \widetilde{D}_t}{1 + r_f} + \sum_{s=t+1}^{T-1} \frac{\tau r_f (\tau r_f)^{s-t} \widetilde{L}^d_s \dots \widetilde{L}^d_{t+1} \widetilde{D}_t}{(1 + r_f)^{s-t+1}} + \sum_{s=t}^{T-1} \sum_{u=t+1}^{s} \frac{\mathrm{E}^Q_t \left[\tau r_f \left(\widetilde{L}^d_s \dots \widetilde{L}^d_u (\tau r_f)^{s-u} \widetilde{CF}^u_u \right) \right]}{(1 + r_f)^{s-t+1}} .$$

This simplifies to (let $\widetilde{L}^d_s \dots \widetilde{L}^d_{t+1} = 1$ if $s = t$)

$$\widetilde{V}^l_t = \widetilde{V}^u_t + \widetilde{D}_t \sum_{s=t}^{T-1} \frac{(\tau r_f)^{s-t+1}}{(1 + r_f)^{s-t+1}} \widetilde{L}^d_s \dots \widetilde{L}^d_{t+1} + \sum_{s=t}^{T-1} \sum_{u=t+1}^{s} \frac{\widetilde{L}^d_s \dots \widetilde{L}^d_u (\tau r_f)^{s+1-u}}{(1 + r_f)^{s+1-u}} \frac{\mathrm{E}^Q_t \left[\widetilde{CF}^u_u \right]}{(1 + r_f)^{u-t}} .$$

Changing summation it yields

$$\widetilde{V}^l_t = \widetilde{V}^u_t + \widetilde{D}_t \sum_{s=t}^{T-1} \frac{(\tau r_f)^{s-t+1}}{(1 + r_f)^{s-t+1}} \widetilde{L}^d_s \dots \widetilde{L}^d_{t+1} + \sum_{u=t+1}^{T-1} \frac{\mathrm{E}^Q_t \left[\widetilde{CF}^u_u \right]}{(1 + r_f)^{u-t}} \sum_{s=u}^{T-1} \frac{\widetilde{L}^d_s \dots \widetilde{L}^d_u (\tau r_f)^{s+1-u}}{(1 + r_f)^{s+1-u}}$$

or after using Thm. 2.2

$$\widetilde{V}^l_t = \widetilde{V}^u_t + \widetilde{D}_t \sum_{s=t}^{T-1} \frac{(\tau r_f)^{s-t+1}}{(1 + r_f)^{s-t+1}} \widetilde{L}^d_s \dots \widetilde{L}^d_{t+1}$$

$$+ \sum_{u=t+1}^{T-1} \left(\sum_{s=u}^{T-1} \frac{\widetilde{L}_s^d \dots \widetilde{L}_u^d (\tau r_f)^{s+1-u}}{(1+r_f)^{s+1-u}} \right) \frac{\mathrm{E}_t \left[\widetilde{CF}_u^u \right]}{(1+k^{E,u})^{u-t}} \,.$$

Changing indices gives Thm. 2.24.

Now let us turn to the case of infinite lifetime (Theorem 2.25) and constant dynamic leverage ratio. In this case

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \widetilde{D}_t \sum_{s=t}^{\infty} \frac{(\tau r_f)^{s-t+1}}{(1+r_f)^{s-t+1}} (\widetilde{L}^d)^{s-t}$$
$$+ \sum_{s=t+1}^{\infty} \left(\sum_{u=s}^{\infty} \frac{(\widetilde{L}^d)^{u+1-s} (\tau r_f)^{u+1-s}}{(1+r_f)^{u+1-s}} \right) \frac{\mathrm{E}_t \left[\widetilde{CF}_s^u \right]}{(1+k^{E,u})^{s-t}} \,.$$

The sum of geometric series gives

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \widetilde{D}_t \frac{\tau r_f}{1+r_f(1-\tau\widetilde{L}^d)} + \frac{\widetilde{L}^d \tau r_f}{1+r_f(1-\tau\widetilde{L}^d)} \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t \left[\widetilde{CF}_s^u \right]}{(1+k^{E,u})^{s-t}} \,.$$

The sum on the right hand side is just $\widetilde{V}_t^u$ and hence

$$\widetilde{V}_t^l = \widetilde{D}_t \frac{\tau r_f}{1+r_f(1-\tau\widetilde{L}^d)} + \widetilde{V}_t^u \left(1 + \frac{\widetilde{L}^d \tau r_f}{1+r_f(1-\tau\widetilde{L}^d)} \right) \,.$$

This finishes the proof. □

5.7 Valuation Formula with Retention Based on Dividends (Thm. 3.9)

Proof. The free cash flows are never lower than the dividend (see Assump. 3.2). From Eq. (3.2) and Def. 3.4 it follows

$$\mathrm{E}_{s-1}^Q \left[\widetilde{A}_s \right] = \mathrm{E}_{s-1}^Q \left[\frac{1}{1-\tau^D} \widetilde{CF}_s^u - Div_s \right] + (1+r_f) \widetilde{A}_{s-1} \,.$$

Using induction and Rule 4, and noting that dividends are deterministic, it follows for $s > t$ that

$$\mathrm{E}_t^Q \left[\widetilde{A}_s \right] = (1+r_f)^{s-t} \widetilde{A}_t + \sum_{v=t+1}^{s} (1+r_f)^{s-v} \left(\frac{\mathrm{E}_t^Q \left[\widetilde{CF}_v^u \right]}{1-\tau^D} - Div_v \right) \,.$$

We plug this term into (3.7). It yields

$$\begin{aligned}
\widetilde{V}_t^l &= \widetilde{V}_t^u + \left(1-\tau^D\right)\widetilde{A}_t + \sum_{s=t}^{n} \frac{\mathrm{E}_t^Q\left[\tau^I r_f\left(1-\tau^D\right)\widetilde{A}_s\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t+1}} \\
&= \widetilde{V}_t^u + \left(1-\tau^D\right)\widetilde{A}_t + \sum_{s=t}^{n} \frac{\tau^I r_f\left(1-\tau^D\right)\left(1+r_f\right)^{s-t}\widetilde{A}_t}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t+1}} \\
&\qquad + \sum_{s=t+1}^{n}\sum_{v=t+1}^{s} \frac{\tau^I r_f\left(1+r_f\right)^{s-v}\left(\mathrm{E}_t^Q\left[\widetilde{CF}_v^u\right]-\left(1-\tau^D\right)Div_v\right)}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t+1}} \\
&= \widetilde{V}_t^u + \tau^I\left(1-\tau^D\right)\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n-t+1}\widetilde{A}_t \\
&\qquad + \sum_{v=t+1}^{n}\sum_{s=v}^{n} \frac{\tau^I r_f\left(1+r_f\right)^{s-v}\left(\mathrm{E}_t^Q\left[\widetilde{CF}_v^u\right]-\left(1-\tau^D\right)Div_v\right)}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t+1}}\,,
\end{aligned}$$

the last row by changing summands. Using geometric sums we get

$$\begin{aligned}
\widetilde{V}_u^l &= \widetilde{V}_t^u + \tau^I\left(1-\tau^D\right)\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n-t+1}\widetilde{A}_t \\
&\quad + \tau^I r_f \sum_{v=t+1}^{n} \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_v^u\right]-\left(1-\tau^D\right)Div_v}{\left(1+r_f\left(1-\tau^I\right)\right)^{v-t}}\left(1+\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n+1-v}\right)\,.
\end{aligned}$$

Lastly, we use Thm. 3.4 and get

$$\begin{aligned}
\widetilde{V}_u^l &= \widetilde{V}_t^u + \tau^I\left(1-\tau^D\right)\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n-t+1}\widetilde{A}_t \\
&+\tau^I r_f \sum_{v=t+1}^{n}\left(\frac{\mathrm{E}_t\left[\widetilde{CF}_v^u\right]}{\left(1+k^{E,u}\right)^{v-t}} - \frac{\left(1-\tau^D\right)Div_v}{\left(1+r_f\left(1-\tau^I\right)\right)^{v-t}}\right)\left(1+\left(\frac{1+r_f}{1+r_f\left(1-\tau^I\right)}\right)^{n+1-v}\right)\,.
\end{aligned}$$

This was to be shown. □

5.8 References

Harrison JM, Kreps DM (1979) Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20(3):381–408

Chapter 6
Sketch of Solutions

Abstract This chapter contains the solutions to the problems. Note that all numerical results were computed using software, and rounding was applied only after the calculations were completed.

6.1 Basic Elements

Solution 1.1

a) We have

$$V_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{1+\kappa_{0,1}} + \frac{CF_2}{\left(1+r_f\right)^2} = \frac{100}{1.10} + \frac{100}{1.05^2} \approx 181.61 \ .$$

The resulting cost of capital is

$$k_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1 + V_1\right]}{V_0} - 1 \approx \frac{100 + \frac{100}{1.05}}{181.61} \approx 7.503\% \ .$$

b) We have

$$V_0 = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{1+k_0} + \frac{CF_2}{(1+k_0)\left(1+r_f\right)} = \frac{100}{1.10} + \frac{100}{1.10 \times 1.05} \approx 177.49 \ .$$

The discount rate is then

$$\kappa_{0,1} = \frac{\mathrm{E}\left[\widetilde{CF}_1\right]}{V_0 - \frac{CF_2}{\left(1+r_f\right)^2}} - 1 \approx \frac{100}{177.49 - \frac{100}{1.05^2}} \approx 15.226\% \ .$$

Solution 1.2 This is straightforward, see Fig. 6.1.

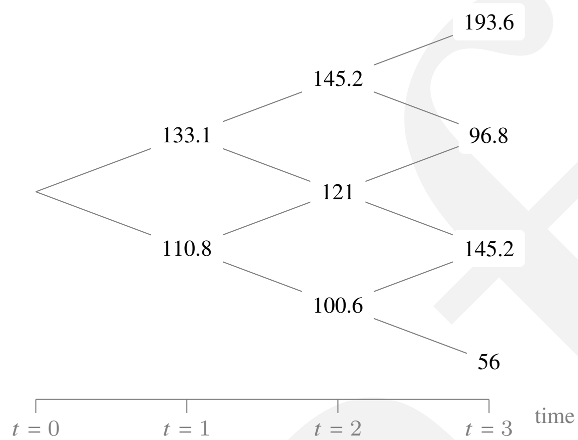

Fig. 6.1 Cash flows in Prob.1.2.

Solution 1.3 The expectation for cash flows from Fig. 1.4 is evaluated as

$$\mathrm{E}_0\left[\mathrm{E}_1^Q\left[\widetilde{CF}_2\right]\right] = \frac{1}{2}\,(0.1\times 145.2 + 0.9\times 121) + \frac{1}{2}\,(0.1\times 121 + 0.9\times 100.6) = 113.03$$

and

$$\mathrm{E}_0^Q\left[\mathrm{E}_1\left[\widetilde{CF}_2\right]\right] = 0.1\left(\frac{1}{2}\times 145.2 + \frac{1}{2}\times 121\right) + 0.9\left(\frac{1}{2}\times 121 + \frac{1}{2}\times 100.6\right) = 113.03\ .$$

Let us turn to Fig. 1.5. If we now change the order of expectation we get

$$\mathrm{E}_0\left[\mathrm{E}_1^Q\left[\widetilde{CF}_2\right]\right] = \frac{1}{2}\,(0.1\times 145.2 + 0.9\times 120) + \frac{1}{2}\,(0.1\times 122 + 0.9\times 100.6) = 112.63$$

and this does not equal

$$\mathrm{E}_0^Q\left[\mathrm{E}_1\left[\widetilde{CF}_2\right]\right] = 0.1\left(\frac{1}{2}\times 145.2 + \frac{1}{2}\times 120\right)$$

$$+ 0.9\left(\frac{1}{2}\times 122 + \frac{1}{2}\times 100.6\right) = 113.43\ .$$

In general, it can be shown that both expectations may be changed if down-up and up-down yield the same cashflows. Otherwise not. The rule is: If the outcome is independent of the actual path, i.e., if ud yields the same as du and uud yields the same as udu and duu, then expectations may be changed.

Solution 1.4 For the conditional expectation of $\widetilde{CF}_3$ given time $t = 2$ we have four possible realizations: up-up, up-down, down-up and down-down. We start with up-up and get

$$\begin{aligned} \mathrm{E}_2\left[\widetilde{CF}_3\right](uu) &= \frac{u^3 + u^2 d}{2} CF_0 \\ &= u^2 CF_0 \\ &= \widetilde{CF}_2(uu)\ . \end{aligned}$$

The other equations for ud, du as well as dd follow analogously,

$$\mathrm{E}_2\left[\widetilde{CF}_3\right] = \widetilde{CF}_2\ .$$

Solution 1.5 We have

$$\mathrm{E}_1\left[\widetilde{CF}_2\right] = \begin{cases} \frac{u+m+d}{3}\,\widetilde{CF}_1(u) & \text{if } \widetilde{CF}_1 \text{ is up,} \\ \frac{u+m+d}{3}\,\widetilde{CF}_1(m) & \text{if } \widetilde{CF}_1 \text{ is middle,} \\ \frac{u+m+d}{3}\,\widetilde{CF}_1(d) & \text{if } \widetilde{CF}_1 \text{ is down.} \end{cases}$$

The required relation holds if

$$u + m + d = 3\ .$$

Jointly with $ud = m^2$ this gives the solution

$$u = \frac{(3-m) + \sqrt{3(1-m)(m+3)}}{2}, \quad d = \frac{(3-m) - \sqrt{3(1-m)(m+3)}}{2}\ .$$

The plot 6.2 below shows (u, m, d) for m from $[0, 1]$. m must be less then 1.

Solution 1.6

a) The values are straightforward: We have $\widetilde{V}_1(u) = \frac{\frac{1}{2}1+\frac{1}{2}2}{1+0\%} = 1.5$ and $\widetilde{V}_1(d) = \frac{\frac{1}{2}4+\frac{1}{2}6}{1+0\%} = 5$. From this we get

$$V_0 = \frac{\frac{1}{2}(1+1.5) + \frac{1}{2}(5+2)}{1+0\%} = 4.75\ .$$

b) Now we can evaluate the unconditional expectations in a straightforward manner,

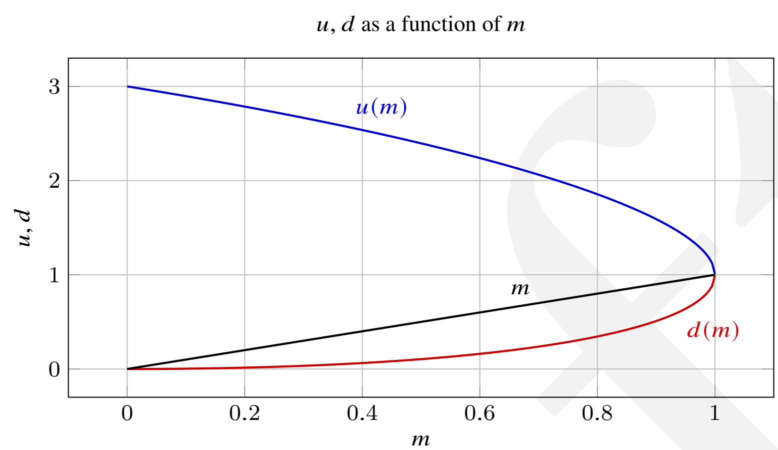

Fig. 6.2 Growth factors (trinomial tree) in Prob.1.7.

$$\mathrm{E}[\widetilde{V}_1] = 3.6, \quad \mathrm{E}[\widetilde{CF}_2] = 3.76\ .$$

From this it follows

$$\frac{\mathrm{E}[\widetilde{CF}_2]}{\mathrm{E}[\widetilde{V}_1]} - 1 = \frac{3.76}{3,6} - 1 \approx 4.44\%\ .$$

We get different values if we use conditional expectations, namely

$$\frac{\mathrm{E}_1[\widetilde{CF}_2]}{\widetilde{V}_1} - 1 = \begin{cases} \frac{1.6}{1.5} - 1 \approx 6.67\% & \text{up} \\ \frac{5.2}{5} - 1 = 4\% & \text{down} \end{cases}$$

The expected value of the two cost of capital is different from 4.44%. Also, $\mathrm{E}[\widetilde{V}_1 + \widetilde{CF}_1] = 5.2$, and hence $\frac{\mathrm{E}[\widetilde{V}_1 + \widetilde{CF}_1]}{V_0} - 1 \approx 9.47\%$.

Solution 1.7 From $s_1 - 1 \geq s_2 - 1 \geq t$ it follows

$$\begin{aligned} \mathrm{E}_t\left[\widetilde{r}_{s_1}\widetilde{r}_{s_2}\right] &= \mathrm{E}_t\left[\mathrm{E}_{s_2}\left[\widetilde{r}_{s_1}\widetilde{r}_{s_2}\right]\right] && \text{by Rule 4} \\ &= \mathrm{E}_t\left[\widetilde{r}_{s_2}\,\mathrm{E}_{s_2}\left[\widetilde{r}_{s_1}\right]\right] && \text{by Rule 5} \\ &= \mathrm{E}_t\left[\widetilde{r}_{s_2}\,\mathrm{E}_{s_2}\left[\underbrace{\mathrm{E}_{s_1-1}\left[\widetilde{r}_{s_1}\right]}_{:=k_{s_1}}\right]\right] && \text{by Rule 4.} \end{aligned}$$

Now using Def. 1.1 and because k_{s_1} is a real number (by assumption) it follows from Rule 2 (Linearity) that

$$\mathrm{E}_t\left[\widetilde{r}_{s_1}\widetilde{r}_{s_2}\right] = k_{s_1}\,\mathrm{E}_t\left[\widetilde{r}_{s_2}\right]\ .$$

Employing Rule 4 (Iterated Expectations) yields

$$\begin{aligned} k_{s_1} &= \mathrm{E}_t\left[\mathrm{E}_{s_1-1}\left[\widetilde{r}_{s_1}\right]\right] \\ &= \mathrm{E}_t\left[\widetilde{r}_{s_1}\right] \end{aligned}$$

which gives the desired result.

Solution 1.8

a) From (1.9) it follows

$$\begin{aligned} \frac{\mathrm{E}_t^Q[\widetilde{V}_T]}{(1+r_f)^{T-t}} &= \frac{\mathrm{E}_t^Q[\mathrm{E}_{T-1}^Q[\widetilde{V}_T]]}{(1+r_f)^{T-t}} \\ &= \frac{\mathrm{E}_t^Q[(1+r_f)\widetilde{V}_{T-1}]}{(1+r_f)^{T-t}} \\ &= \frac{\mathrm{E}_t^Q[\widetilde{V}_{T-1}]}{(1+r_f)^{(T-1)-t}} \\ &\dots \\ &= \widetilde{V}_t \end{aligned}$$

b) Letting $T \to \infty$, transversality (1.12) implies that the right hand side goes to zero. Hence $\widetilde{V}_t = 0$.

The statement in subproblem c. also follows directly from (1.13).

Solution 1.9 k is constant. We use Thm. 1.1 as well as (1.7) (we use $g = 0$ in our proof)

$$\begin{aligned} \widetilde{V}_t &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t\left[\widetilde{CF}_s\right]}{(1+k)^{s-t}} \\ &= \sum_{s=t+1}^{\infty} \frac{(1+g)^{s-t}\,\widetilde{CF}_t}{(1+k)^{s-t}} \\ &= \frac{1+g}{k-g}\,\widetilde{CF}_t\ . \end{aligned}$$

Now

$$\widetilde{V}_{t+1} = \begin{cases} u\,\widetilde{V}_t & \text{if up,} \\ d\,\widetilde{V}_t & \text{if down} \end{cases}$$

is an immediate consequence of the fact that $\widetilde{CF}_t$ follows this pattern. The second part of the solution follows from $\frac{u}{2} + \frac{d}{2} = 1$.

Solution 1.10 We know that

$$\widetilde{V}_t = \frac{\mathrm{E}_t^Q[\widetilde{V}_{t+1} + \widetilde{CF}_{t+1}]}{1+r_f}$$

holds. Now adding $C(1+r_f)^t$ to both sides

$$\widetilde{V}_t + C(1+r_f)^t = \frac{\mathrm{E}_t^Q[\widetilde{V}_{t+1} + C(1+r_f)^{t+1} + \widetilde{CF}_{t+1}]}{1+r_f}$$

and replacing $\widetilde{V}_t^* := \widetilde{V}_t + C(1+r_f)^t$ gives the desired result

$$\widetilde{V}_t^* = \frac{\mathrm{E}_t^Q[\widetilde{V}_{t+1}^* + \widetilde{CF}_{t+1}]}{1+r_f} .$$

Both $\widetilde{V}$ as well as $\widetilde{V}^*$ satisfy the Fundamental Theorem.

Note that $\widetilde{V}^*$ does not satisfy the transversality condition if $\widetilde{V}$ did: We have

$$\lim_{T\to\infty} \frac{\mathrm{E}_t^Q[\widetilde{V}_T + C(1+r_f)^T]}{(1+r_f)^{T-t}} = \lim_{T\to\infty} \frac{\mathrm{E}_t^Q[\widetilde{V}_T]}{(1+r_f)^{T-t}} + C(1+r_f)^t$$

and this cannot be zero if C is arbitrary.

Solution 1.11

a) We denote the probabilities by $P(u), P(d)$ and $Q(u), Q(d)$. Since the expectation under P is zero, we have

$$P(u)\widetilde{CF}_1(u) + \big(1 - P(u)\big)\widetilde{CF}_1(d) = 0 .$$

Now assume the same holds for the risk-neutral probabilities Q. By subtracting one equation from the other we get

$$0 = \big(P(u) - Q(u)\big) \times \big(\widetilde{CF}_1(u) - \widetilde{CF}_1(d)\big) ,$$

and this equation can only be satisfied if either the probabilities are equal or the cash flows are identical in both states. But this is precisely what we excluded. Hence, the Q-expected value of the cash flow cannot be zero.

This implies that the value of the cash flow, defined as

$$V_0 = \frac{\mathrm{E}^Q[\widetilde{CF}_1]}{1+r_f} ,$$

must also be different from zero.

b) We can now evaluate the cost of capital. Since the denominator is nonzero, the expression is well defined:

$$k = \frac{\overbrace{\mathrm{E}[\widetilde{CF}_1]}^{=0}}{V_0} - 1 = -100\% ,$$

which was to be shown.

Solution 1.12 We begin with

$$\begin{aligned} V_0 &= \frac{\mathrm{E}_0\left[\widetilde{CF}_1\right]}{1+r_f} + \mathrm{Cov}\left[m_1, \widetilde{CF}_1\right] && \text{see (1.14)} \\ 1+r_f &= \frac{\mathrm{E}_0\left[\widetilde{CF}_1\right]}{V_0} + (1+r_f)\,\mathrm{Cov}\left[m_1, \frac{\widetilde{CF}_1}{V_0}\right] \\ r_f &= \mathrm{E}_0\left[\frac{\widetilde{CF}_1}{V_0} - 1\right] + (1+r_f)\,\mathrm{Cov}\left[m_1, \frac{\widetilde{CF}_1}{V_0} - 1\right] \\ r_f - \mathrm{E}_0\left[r\right] &= (1+r_f)\,\mathrm{Cov}\left[m_1, r\right] \ . \end{aligned}$$

This equation holds for all assets. Setting $\widetilde{CF}_1 = m_1$ implies $r = r_m$ and we get

$$r_f - \mathrm{E}_0\left[r_m\right] = (1+r_f)\,\mathrm{Cov}\left[m_1, r_m\right] \ .$$

Dividing both equations results in

$$\mathrm{E}_0\left[r\right] - r_f = \left(\mathrm{E}_0\left[r_m\right] - r_f\right)\frac{\mathrm{Cov}\left[m_1, r\right]}{\mathrm{Cov}\left[m_1, r_m\right]} \ .$$

If we now divide the right fraction by $V_0(m)$ and use $\mathrm{Cov}\left[\frac{m_1}{V_0(m)}, r_m\right] = \mathrm{Cov}\left[\frac{m_1}{V_0(m)} - 1, r_m\right]$ we get the assertion.

6.2 Corporate Income Tax

Solution 2.1 Assuming iid, the $\mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u]$ is constant over time (the same holds under P), hence

$$\forall t = 0, 1, \ldots \qquad \mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u] = \mathrm{E}_0^Q[\widetilde{CF}_1^u] =: \text{const.} \ . \tag{6.1}$$

Then, from the valuation equation it follows that the firm value is constant,

$$\begin{aligned} V_t^u &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t^Q[\widetilde{CF}_s^u]}{(1+r_f)^{s-t}} \\ &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_0^Q[\widetilde{CF}_1^u]}{(1+r_f)^{s-t}} \\ &= \mathrm{E}_0^Q[\widetilde{CF}_1^u] \sum_{s=t+1}^{\infty} \frac{1}{(1+r_f)^{s-t}} \\ &= \frac{\mathrm{E}_0^Q[\widetilde{CF}_1^u]}{r_f} \ . \end{aligned}$$

Hence, cost of capital is deterministic and constant as well,

$$k_t = \frac{\mathrm{E}_0[\widetilde{CF}_1^u] + V_{t+1}}{V_t^u} - 1 = \frac{\mathrm{E}_0[\widetilde{CF}_1^u] + \frac{\mathrm{E}_0^Q[\widetilde{CF}_1^u]}{r_f}}{\frac{\mathrm{E}_0^Q[\widetilde{CF}_1^u]}{r_f}} - 1 = r_f \frac{\mathrm{E}_0[\widetilde{CF}_1^u]}{\mathrm{E}_0^Q[\widetilde{CF}_1^u]} .$$

Since the cost of capital k_t is different from the risk-free rate we conclude $\mathrm{E}_0[\widetilde{CF}_1^u] \neq \mathrm{E}_0^Q[\widetilde{CF}_1^u]$.

We next show that the discount factor k_t' is different from the cost of capital k_t. To this end we assume otherwise and use the equation above for k_t,

$$\begin{aligned}
\frac{\mathrm{E}_t[\widetilde{CF}_{t+1}^u]}{1+k_t} &\stackrel{?}{=} \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u]}{1+r_f} \\
\frac{\mathrm{E}_t[\widetilde{CF}_{t+1}^u]}{1 + r_f \frac{\mathrm{E}_0[\widetilde{CF}_1^u]}{\mathrm{E}_0^Q[\widetilde{CF}_1^u]}} &\stackrel{?}{=} \frac{\mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u]}{1+r_f} \\
(1+r_f)\,\mathrm{E}_t[\widetilde{CF}_{t+1}^u] &\stackrel{?}{=} \mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u] + r_f\,\mathrm{E}_t[\widetilde{CF}_{t+1}^u] \\
\mathrm{E}_t[\widetilde{CF}_{t+1}^u] &\stackrel{?}{=} \mathrm{E}_t^Q[\widetilde{CF}_{t+1}^u]
\end{aligned}$$

and we have showed above that this equation cannot hold. Hence, discount rates and cost of capital are different in this case.

Solution 2.2 Figure 6.3 shows the individual growth rates. It is straightforward to see that at each node (state) they add up to 20% which corresponds to $2 \times g$ because each future state has a probability of 50%. We added $CF_0 = \frac{\mathrm{E}[\widetilde{CF}_1^u]}{1+g} = \frac{100}{1.1} \approx 90.91$ for the first node at $t = 0$.

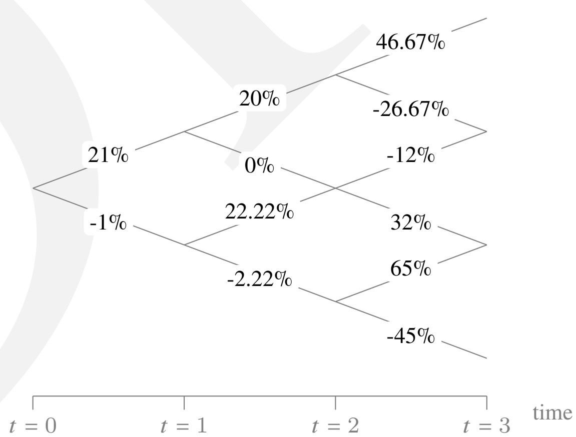

Fig. 6.3 Growth rates in Prob.2.2.

Solution 2.3 From the Fundamental Theorem we have

$$\begin{aligned}
\widetilde{V}_t^u &= \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u + \widetilde{V}_{t+1}^u\right]}{1+r_f} \\
\widetilde{CF}_t^u \frac{1+g}{k^{E,u}-g} &= \frac{\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u + \widetilde{CF}_{t+1}^u \frac{1+g}{k^{E,u}-g}\right]}{1+r_f} \\
\mathrm{E}_t^Q\left[\widetilde{CF}_{t+1}^u\right] &= \underbrace{\frac{(1+g)\left(1+r_f\right)}{1+k^{E,u}}}_{=:1+g^Q} \widetilde{CF}_t^u \,.
\end{aligned}$$

Solution 2.4

a) First we have

$$\mathrm{E}_t\left[\widetilde{CF}_s^u\right] = (s-t)C + \widetilde{CF}_t^u$$

by induction. But then

$$\begin{aligned}
\widetilde{V}_t^u &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{(1+k^{E,u})^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{\widetilde{CF}_t^u + (s-t)C}{(1+k^{E,u})^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{\widetilde{CF}_t^u}{(1+k^{E,u})^{s-t}} + \sum_{s=t+1}^{\infty} \frac{(s-t)C}{(1+k^{E,u})^{s-t}} \\
&= \frac{\widetilde{CF}_t^u}{k^{E,u}} + \frac{1+k^{E,u}}{(k^{E,u})^2} C
\end{aligned}$$

using an algebraic formula.[1] In particular, the price-dividend ratio is not deterministic any more. Compare this to Prob.2.14.

---

[1] Note that

$$\sum_{s=1}^{\infty} \frac{1}{(1+x)^s} = \frac{1}{x}.$$

Differentiating this relation yields

$$\sum_{s=1}^{\infty} \frac{-s}{(1+x)^{s+1}} = -\frac{1}{x^2}$$

or after multiplying by $-(1+x)$

$$\sum_{s=1}^{\infty} \frac{s}{(1+x)^s} = \frac{1+x}{x^2} \,.$$

b) We have

$$\begin{aligned}\mathrm{E}_t\left[\widetilde{V}^u_{t+1}\right] &= \mathrm{E}_t\left[\frac{\widetilde{CF}^u_{t+1}}{k^{E,u}} + \frac{1+k^{E,u}}{(k^{E,u})^2}\,C\right] \\ &= \frac{\widetilde{CF}^u_t + C}{k^{E,u}} + \frac{1+k^{E,u}}{(k^{E,u})^2}\,C \\ &= \widetilde{V}^u_t + \frac{C}{k^{E,u}}\end{aligned}$$

and hence the expected capital gains rate is not zero.

c) We use

$$\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+k^{E,u}} = \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+r_f} = \widetilde{V}^u_t\ .$$

Employing the above result this implies

$$\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1} + \frac{\widetilde{CF}^u_{t+1}}{k^{E,u}} + \frac{1+k^{E,u}}{(k^{E,u})^2}\,C\right]}{1+k^{E,u}} = \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1} + \frac{\widetilde{CF}^u_{t+1}}{k^{E,u}} + \frac{1+k^{E,u}}{(k^{E,u})^2}\,C\right]}{1+r_f}$$

and after rearranging and multiplying by $\frac{k^{E,u}}{1+k^{E,u}}$,

$$\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] + \frac{1}{k^{E,u}}\,C}{1+k^{E,u}} = \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right] + \frac{1}{k^{E,u}}\,C}{1+r_f}\ . \tag{6.2}$$

Rearranging once again gives

$$\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right] = \frac{1+r_f}{1+k^{E,u}}\,\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] + \frac{1+r_f}{k^{E,u}}\left(\frac{1}{1+k^{E,u}} - \frac{1}{1+r_f}\right) C$$

which is, after using the assumption,

$$\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right] = \frac{1+r_f}{1+k^{E,u}}\left(\widetilde{CF}^u_t + C\right) + \frac{C}{k^{E,u}}\left(\frac{1+r_f}{1+k^{E,u}} - 1\right)\ .$$

Summarizing the terms with C gives the required result.
Now consider Thm. 2.3. Since C does not cancel in (6.2) this intermediate result shows that our assumption will not yield a similar statement as in Thm. 2.3. We have

$$\frac{C}{k^{E,u}(1+k^{E,u})} \neq \frac{C}{k^{E,u}\left(1+r_f\right)} \quad\Longrightarrow\quad \frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{1+k^{E,u}} \neq \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f}\ .$$

Hence, using this assumption about cash flows we can somehow use cost of capital as discount rates but we have to add a correction term due to the fact that $C \neq 0$.

Solution 2.5

a) First we have by rule 4

$$\begin{aligned}
\mathrm{E}_t\left[\widetilde{CF}^u_{t+2}\right] &= \mathrm{E}_t\left[\mathrm{E}_{t+1}\left[\widetilde{CF}^u_{t+2}\right]\right] \\
&= \mathrm{E}_t\left[\widetilde{CF}^u_{t+1} + X_{t+1}\right] \\
&= \mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right] \\
&= \widetilde{CF}^u_t + X_t
\end{aligned}$$

and hence for $s > t$

$$\mathrm{E}_t\left[\widetilde{CF}^u_s\right] = \widetilde{CF}^u_t + X_t \ .$$

The value of the company is given by

$$\begin{aligned}
\widetilde{V}^u_t &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t\left[\widetilde{CF}^u_s\right]}{(1+k^{E,u})^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{\widetilde{CF}^u_t + X_t}{(1+k^{E,u})^{s-t}} \\
&= \frac{\widetilde{CF}^u_t}{k^{E,u}} + \frac{X_t}{k^{E,u}} \ .
\end{aligned}$$

Since $\widetilde{CF}^u_t$ and X_t are uncorrelated the variance of the firm is greater than the variance of the cash flows (if $k^{E,u} < 100\%$),

$$\begin{aligned}
\mathrm{Var}\left[\widetilde{V}^u_t\right] &= \mathrm{Var}\left[\frac{\widetilde{CF}^u_t}{k^{E,u}} + \frac{X_t}{k^{E,u}}\right] \\
&= \frac{\mathrm{Var}\left[\widetilde{CF}^u_t\right]}{(k^{E,u})^2} + \frac{\mathrm{Var}\left[X_t\right]}{(k^{E,u})^2} \\
&> \frac{\mathrm{Var}\left[\widetilde{CF}^u_t\right]}{(k^{E,u})^2} > \mathrm{Var}\left[\widetilde{CF}^u_t\right] \ .
\end{aligned}$$

b) Now

$$
\begin{aligned}
\mathrm{E}_t\left[\widetilde{V}^u_{t+1}\right] &= \mathrm{E}_t\left[\frac{\widetilde{CF}^u_{t+1}}{k^{E,u}} + \frac{X_{t+1}}{k^{E,u}}\right] \\
&= \frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{k^{E,u}} \\
&= \frac{\widetilde{CF}^u_t + X_t}{k^{E,u}} = \widetilde{V}^u_t
\end{aligned}
$$

and the expected capital gains rate is zero.

c) This is harder to show and we closely follow the proof of Thm. 2.3.

$$
\begin{aligned}
\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+k^{E,u}} &= \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+r_f} = \widetilde{V}^u_t \\
\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1} + \frac{\widetilde{CF}^u_{t+1}+X_{t+1}}{k^{E,u}}\right]}{1+k^{E,u}} &= \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1} + \frac{\widetilde{CF}^u_{t+1}+X_{t+1}}{k^{E,u}}\right]}{1+r_f} \\
\frac{(1+\frac{1}{k^{E,u}})\,\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{1+k^{E,u}} &= \frac{(1+\frac{1}{k^{E,u}})\,\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f} \\
\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]}{1+k^{E,u}} &= \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1}\right]}{1+r_f}
\end{aligned}
$$

which is the first part of the claim.
The second part follows from

$$
\begin{aligned}
\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+k^{E,u}} &= \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+1} + \widetilde{V}^u_{t+1}\right]}{1+r_f} = \widetilde{V}^u_t \\
\frac{\mathrm{E}_t\left[\widetilde{V}^u_{t+1}\right]}{1+k^{E,u}} &= \frac{\mathrm{E}^Q_t\left[\widetilde{V}^u_{t+1}\right]}{1+r_f} \\
\frac{\mathrm{E}_t\left[\frac{\mathrm{E}_{t+1}\left[\widetilde{CF}^u_{t+2}+\widetilde{V}^u_{t+2}\right]}{1+k^{E,u}}\right]}{1+k^{E,u}} &= \frac{\mathrm{E}^Q_t\left[\frac{\mathrm{E}^Q_{t+1}\left[\widetilde{CF}^u_{t+2}+\widetilde{V}^u_{t+2}\right]}{1+r_f}\right]}{1+r_f} \\
\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+2} + \widetilde{V}^u_{t+2}\right]}{(1+k^{E,u})^2} &= \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+2} + \widetilde{V}^u_{t+2}\right]}{\left(1+r_f\right)^2} \\
\frac{\mathrm{E}_t\left[\widetilde{CF}^u_{t+2} + \frac{\widetilde{CF}^u_{t+2}+X_{t+2}}{k^{E,u}}\right]}{(1+k^{E,u})^2} &= \frac{\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+2} + \frac{\widetilde{CF}^u_{t+2}+X_{t+2}}{k^{E,u}}\right]}{\left(1+r_f\right)^2}
\end{aligned}
$$

and since $\mathrm{E}_{t+1}\left[X_{t+2}\right] = \mathrm{E}^{Q}_{t+1}\left[X_{t+2}\right] = 0$,

$$\frac{\left(1+\frac{1}{k^{E,u}}\right)\mathrm{E}_t\left[\widetilde{CF}^u_{t+2}\right]}{(1+k^{E,u})^2} = \frac{\left(1+\frac{1}{k^{E,u}}\right)\mathrm{E}^Q_t\left[\widetilde{CF}^u_{t+2}\right]}{\left(1+r_f\right)^2}$$

which is almost the claim.

Solution 2.6 We start with (2.20) and get for $s > t$

$$\mathrm{E}^Q_t\left[\widetilde{D}_s\right] = \frac{\mathrm{E}^Q_t\left[\widetilde{I}_{s+1} + \widetilde{D}_{s+1} + \widetilde{Pr}_{s+1}\right]}{1+r_f} .$$

At time $s+1$ we similarly have

$$\mathrm{E}^Q_t\left[\widetilde{D}_{s+1}\right] = \frac{\mathrm{E}^Q_t\left[\widetilde{I}_{s+2} + \widetilde{D}_{s+2} + \widetilde{Pr}_{s+2}\right]}{1+r_f} .$$

Plugging this into the first equation gives

$$\begin{aligned}\mathrm{E}^Q_t\left[\widetilde{D}_s\right] &= \frac{\mathrm{E}^Q_t\left[\widetilde{I}_{s+1} + \frac{\mathrm{E}^Q_t\left[\widetilde{I}_{s+2}+\widetilde{D}_{s+2}+\widetilde{Pr}_{s+2}\right]}{1+r_f} + \widetilde{Pr}_{s+1}\right]}{1+r_f} \\ &= \frac{\mathrm{E}^Q_t\left[\widetilde{I}_{s+1} + \widetilde{Pr}_{s+1}\right]}{1+r_f} + \frac{\mathrm{E}^Q\left[\widetilde{I}_{s+2} + \widetilde{Pr}_{s+2}\right]}{(1+r_f)^2} + \frac{\mathrm{E}^Q_t\left[\widetilde{D}_{s+2}\right]}{(1+r_f)^2}\end{aligned}$$

Continuing this approach we get for $s = t$

$$\mathrm{E}^Q_t\left[\widetilde{D}_t\right] = \sum_{s=t+1}^{T} \frac{\mathrm{E}^Q_t\left[\widetilde{I}_s + \widetilde{Pr}_s\right]}{(1+r_f)^{s-t}} + \frac{\mathrm{E}^Q_t\left[\widetilde{D}_T\right]}{(1+r_f)^{T-t}} .$$

Financing is always such that $\widetilde{D}_T = 0$ and $D_t = \mathrm{E}^Q_t\left[\widetilde{D}_t\right]$ (Rule 5, Known Factor). This proves the claim.

Solution 2.7 The tax shield satisfies

$$\widetilde{V}^l_0 - \widetilde{V}^u_0 = \sum_{t=0}^{2} \frac{\tau r_f\,\mathrm{E}^Q\left[\widetilde{D}_t\right]}{\left(1+r_f\right)^t} .$$

Hence, we have to evaluate the expectations $\mathrm{E}^Q\left[\widetilde{D}_t\right]$. We get

$$\mathrm{E}^Q\left[D_0\right] = 100 ,$$

since D_0 is deterministic. It follows that

$$\mathrm{E}^Q\left[\widetilde{D}_1\right] = 120 \times 0.25 + 110 \times 0.75 = 112.5 .$$

At time $t = 2$ we get similarly

$$\mathrm{E}^Q\left[\widetilde{D}_2\right] = 150 \times 0.25^2 + 145 \times 2 \times 0.25 \times 0.75 + 100 \times 0.75^2 = 120 .$$

Using both equations gives us

$$\begin{aligned}\widetilde{V}_0^l - \widetilde{V}_0^u &= \frac{0.1 \times 0.5 \times 100}{1 + 0.1} + \frac{0.1 \times 0.5 \times 112.5}{(1 + 0.1)^2} + \frac{0.1 \times 0.5 \times 120}{(1 + 0.1)^3} \\ &\approx 13.702 .\end{aligned}$$

Solution 2.8 From (2.14) we have

$$\begin{aligned}\widetilde{V}_t^l &= \widetilde{V}_t^u + \sum_{s=t}^{T-1} \frac{\tau r_f \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s+1-t}} \\ &= \widetilde{V}_t^u + \sum_{s=t}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s-t}} \left(1 - \frac{1}{1 + r_f}\right)\end{aligned}$$

and from this

$$\begin{aligned}\widetilde{V}_t^l &= \widetilde{V}_t^u + \sum_{s=t}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s-t}} - \sum_{s=t}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s+1-t}} \\ &= \widetilde{V}_t^u + \tau \widetilde{D}_t + \sum_{s=t+1}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s-t}} - \sum_{s=t}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s+1-t}} \\ &= \widetilde{V}_t^u + \tau \widetilde{D}_t + \sum_{s=t}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_{s+1}\right]}{(1 + r_f)^{s+1-t}} - \sum_{s=t}^{T-1} \frac{\tau \, \mathrm{E}_t^Q\left[\widetilde{D}_s\right]}{(1 + r_f)^{s+1-t}}\end{aligned}$$

which was to be shown (note that $\widetilde{D}_T = 0$).

Solution 2.9 From (2.25) it follows (since the firm is not over-indebted)

$$\begin{aligned}(1 + r_f)\widetilde{D}_t\big(Q(u) + Q(d)\big) &= (1 + c)\widetilde{D}_t Q(u) + \left(\widetilde{V}_{t+1}^u(d) + \widetilde{CF}_{t+1}^u(d)\right) Q(d) \\ (1 + r_f)\widetilde{D}_t Q(u) &\geq (1 + c)\widetilde{D}_t Q(u) + \big(\widetilde{D}_{t+1}(d) + \widetilde{CF}_{t+1}^u(d) \\ &\qquad - (1 + r_f)\widetilde{D}_t\big) Q(d) \\ (r_f - c)\widetilde{D}_t Q(u) &\geq \left(\widetilde{D}_{t+1}(d) + \widetilde{CF}_{t+1}^u(d) - (1 + r_f)\widetilde{D}_t\right) Q(d)\end{aligned}$$

and therefore

$$r_f - c \geq \frac{Q(d)}{\widetilde{D}_t Q(u)} \left(\widetilde{CF}_{t+1}^u(d) - (1 + r_f)\widetilde{D}_t + \widetilde{D}_{t+1}(d)\right) .$$

If now $c \geq r_f$ holds, then it must be that $\widetilde{CF}_{t+1}^u(d) - (1 + r_f)\widetilde{D}_t + \widetilde{D}_{t+1}(d) < 0$.

Solution 2.10 Using (2.26) we get

$$l_1 = \frac{dk^{E,u}}{1 + r_f - d}$$
$$= \frac{0.7 \times 0.2}{1 + 0.1 - 0.7} = 0.35 \ .$$

Illiquidity arises once the leverage ratio exceeds 35%.

In order to determine the coupon under a complete transfer of assets we use (2.27):

$$c = r_f + \frac{1 + r_f - u(1 + k^{E,u})}{k^{E,u}}$$
$$= 0.1 + \frac{1 + 0.1 - 1.1 \times (1 + 0.2)}{0.2} = -100\%.$$

The threshold l_2 is given by

$$l_2 = \frac{1 + k^{E,u}}{1 + r_f} d = \frac{1 + 0.2}{1 + 0.1} 0.7 \approx 76.36\% \ .$$

The transfer rate $\alpha = \frac{l - l_1}{l_2 - l_1} \frac{1 - l_2}{1 - l}$ is shown in Fig. 6.4.

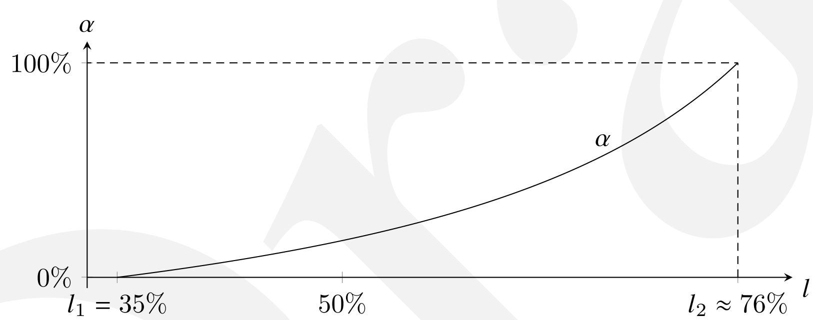

Fig. 6.4 Transfer rate α in Prob. 2.10.

Solution 2.11 We have

$$l(1 + r_f) = (1 + c_t)\, l\, Q(u) + d(1 + k^{E,u})\, Q(d) \ .$$

and using (2.6) as well as (2.26) we get

$$\begin{aligned}
\frac{dk^{E,u}}{1+r_f-d}(1+r_f) &= (1+c_t)\frac{dk^{E,u}}{1+r_f-d}Q(u) + d(1+k^{E,u})Q(d)\\
(1+c_t)k^{E,u}Q(u) &= k^{E,u}(1+r_f) - (1+k^{E,u})(1+r_f-d)Q(d)\\
Q(u)\Big((1+c_t)k^{E,u} - (1+k^{E,u})(1+r_f-d)\Big) &= k^{E,u}(1+r_f) - (1+k^{E,u})(1+r_f-d)\\
\frac{\frac{1+r_f}{1+k^{E,u}}-d}{u-d}\Big((1+c_t)k^{E,u} - (1+k^{E,u})(1+r_f-d)\Big) &= -1-r_f+d+k^{E,u}d\\
(1+c_t)k^{E,u} - (1+k^{E,u})(1+r_f-d) &= (-1-r_f+d+k^{E,u}d)\,\frac{u-d}{\frac{1+r_f}{1+k^{E,u}}-d}\\
(1+c_t)k^{E,u} &= (1+k^{E,u})\big((1+r_f-d)-(u-d)\big)\\
c_t &= \left(1+\frac{1}{k^{E,u}}\right)(1+r_f-u)-1\\
c_t &= \frac{1+r_f-u}{k^{E,u}}+r_f-u
\end{aligned}$$

and this is the first claim. That the coupon is smaller then r_f follows from (2.7):

$$\frac{1+r_f}{1+k^{E,u}} < u \quad\Longrightarrow\quad 1+r_f-u(1+k^{E,u}) < 0\,.$$

Solution 2.12 We have

$$\begin{aligned}
\mathrm{E}_t\left[\widetilde{CF}^l_{t+1}\right] &= \mathrm{E}_t\left[\widetilde{CF}^u_{t+1}+\tau r_f\widetilde{D}_t\right]\\
&= \mathrm{E}_t\left[\widetilde{CF}^u_{t+1}\right]+\tau r_f\widetilde{D}_t\\
&= (1+g_t)\,\widetilde{CF}^u_t+\tau r_f\widetilde{D}_t\\
&\neq (1+g_t)\underbrace{\left(\widetilde{CF}^u_t+\tau r_f\widetilde{D}_{t-1}\right)}_{=\widetilde{CF}^l_t}\,.
\end{aligned}$$

For martingale-like levered cash flows debt $\widetilde{D}_{t-1}$ must increase with the growth rate

$$\widetilde{D}_t = (1+g_t)\widetilde{D}_{t-1} = \ldots = (1+g_t)\ldots(1+g_1)D_0$$

and then deterministic.

Solution 2.13 From (2.14) we have

$$\begin{aligned}
\widetilde{V}^l_t &= \widetilde{V}^u_t + \sum_{s=t}^{T-1}\frac{\tau r_f\,\mathrm{E}^Q_t\left[\widetilde{D}_s\right]}{(1+r_f)^{s+1-t}}\\
&= \widetilde{V}^u_t + \sum_{s=t}^{T-1}\frac{\tau r_f\,\mathrm{E}^Q_t\left[D_t\right]}{(1+r_f)^{s+1-t}}
\end{aligned}$$

$$= \widetilde{V}_t^u + \tau r_f D_t \sum_{s=t}^{T-1} \frac{1}{(1+r_f)^{s+1-t}}$$

and from this the assertion using the formula of perpetual rent.

Solution 2.14 In the infinite example the value of the firm can be rearranged as follows

$$\begin{aligned}
\widetilde{V}_t^l &= \widetilde{V}_t^u + \sum_{s=t+1}^{\infty} \frac{\tau r_f D_s}{\left(1+r_f\right)^{s-t}} \\
&= \frac{\widetilde{CF}_t^u}{k^{E,u}} + \tau D_t \ .
\end{aligned}$$

The dividend-price ratio of the levered firm is

$$\begin{aligned}
d_t^l &= \frac{\widetilde{CF}_t^l}{\widetilde{V}_t^l} \\
&= \frac{\widetilde{CF}_t^u + \tau r_f D_{t-1}}{\widetilde{V}_t^u + \tau D_t} \\
&= k^{E,u} \frac{\widetilde{V}_t^u + \frac{r_f}{k^{E,u}} \tau D_{t-1}}{\widetilde{V}_t^u + \tau D_{t-1}} \ .
\end{aligned}$$

To verify that this dividend ratio is stochastic we first realize that $\frac{r_f}{k^{E,u}} \neq 1$ since the cash flows are stochastic. Then, the function

$$f(x) := \frac{x + \frac{r_f}{k^{E,u}} \tau D_{t-1}}{x + \tau D_{t-1}}$$

is strictly monotone and hence $f(\widetilde{V}_t^u)$ will again be a random variable, i.e., if for two different states $\widetilde{V}_t^u(\omega)$ is different from $\widetilde{V}_t^u(\omega')$, it will be true that $d_t^l(\omega) \neq d_t^l(\omega')$ and hence the dividend-price ratio is stochastic.

Solution 2.15

a) We start with the value of the levered firm at $t = 0$,

$$\begin{aligned}
V_0^l &= \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+WACC} + \frac{\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1+WACC)^2} + \frac{\mathrm{E}\left[\widetilde{CF}_3^u\right]}{(1+WACC)^3} \\
&= \frac{100}{1+0.18} + \frac{110}{(1+0.18)^2} + \frac{121}{(1+0.18)^3} \approx 237.39 \ .
\end{aligned}$$

The value at $t = 1$ depends on the state of nature. We get

$$\widetilde{V}_1^l = \frac{\mathrm{E}_1\left[\widetilde{CF}_2^u\right]}{1+WACC} + \frac{\mathrm{E}_1\left[\widetilde{CF}_3^u\right]}{(1+WACC)^2}$$
$$= \begin{cases} \frac{121}{1+0.18} + \frac{133.1}{(1+0.18)^2} \approx 198.13 & \text{if up,} \\ \frac{99}{1+0.18} + \frac{108.9}{(1+0.18)^2} \approx 162.11 & \text{if down.} \end{cases}$$

b) This debt schedule implies the following leverage ratios:

$$l_0 = \frac{D_0}{V_0^l} = \frac{50}{237.39} \approx 21.06\%$$

and

$$\widetilde{l}_1 = \frac{\widetilde{D}_1}{\widetilde{V}_1^l} = \begin{cases} \frac{60}{198.13} \approx 30.28\% & \text{if up,} \\ \frac{40}{162.11} \approx 24.67\% & \text{if down.} \end{cases}$$

The Miles-Ezzell formula must not be applied.

c) The WACC textbook formula will give us the cost of equity of the levered firm since

$$\widetilde{WACC}_1(u) = \widetilde{k}_1^{E,l}(u)\ \left(1-\widetilde{l}_1(u)\right) + r_f\,(1-\tau)\,\widetilde{l}_1(u)$$
$$0.18 \approx \widetilde{k}_1^{E,l}(u) \times (1-0.3028) + 0.1 \times (1-0.5) \times 0.3028$$

implying

$$\widetilde{k}_1^{E,l}(u) \approx 23.65\%\ .$$

Analogously one gets

$$\widetilde{k}_1^{E,l}(d) \approx 22.26\%\ .$$

The weighted average cost of capital type 1 can be evaluated using the TCF textbook formula

$$\widetilde{k}_1^{\varnothing}(u) = \widetilde{k}_1^{E,l}(u)\ \left(1-\widetilde{l}_1(u)\right) + r_f\,\widetilde{l}_1(u)$$
$$\widetilde{k}_1^{\varnothing}(u) \approx 0.2365 \times (1-0.3028) + 0.1 \times 0.3028 \approx 19.52\%$$

and again

$$\widetilde{k}_1^{\varnothing}(d) = \widetilde{k}_1^{E,l}(d)\ \left(1-\widetilde{l}_1(d)\right) + r_f\,\widetilde{l}_1(d)$$
$$\widetilde{k}_1^{\varnothing}(d) \approx 0.2226 \times (1-0.2467) + 0.1 \times 0.2467 \approx 19.24\%\ .$$

Solution 2.16 We have

$$\text{WACC} - k^{E,u}(1-\tau l_0) = (1+\text{WACC}) - \left(1 + k^{E,u}(1-\tau l)\right)$$
$$= (1+k^{E,u})\left(1 - \frac{\tau r_f}{1+r_f}\,l\right) - \left(1 + k^{E,u}(1-\tau l)\right)$$

$$= \frac{(k^{E,u} - r_f)\tau l}{1 + r_f} > 0$$

which was to be shown.

Solution 2.17 In this case we have

$$\widetilde{V}_t^l = \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t\left[\widetilde{CF}_s^u\right]}{(1 + WACC)^{s-t}} = \sum_{s=t+1}^{\infty} \frac{\widetilde{CF}_t^u}{(1 + WACC)^{s-t}} = \frac{\widetilde{CF}_t^u}{WACC} .$$

Since

$$\widetilde{V}_t^u = \frac{\widetilde{CF}_t^u}{k^{E,u}}$$

using the Miles-Ezzell formula gives

$$\widetilde{V}_t^l = \frac{\widetilde{CF}_t^u}{k^{E,u}} \frac{k^{E,u}}{(1 + k^{E,u})\left(1 - \frac{\tau r_f}{1+r_f} l\right) - 1}$$

which gives the result after some rearranging.

Solution 2.18

a) First, since

$$\widetilde{CF}_{t+1}^u = CF_0^u + \varepsilon_1 + \ldots + \varepsilon_t$$

and using the hint we know that the cash flows are normally distributed. They have expectation CF_0^u and variance t.

b) This is easy, since

$$\widetilde{V}_t^u = \frac{\widetilde{CF}_t^u}{k^{E,u}}$$

the value of the firm is normally distributed and has expectation $\frac{CF_0^u}{k^{E,u}}$ and variance $\frac{t}{(k^{E,u})^2}$.

c) For the book value we have using (5.8) and the assumptions on the past of the firm

$$\underline{\widetilde{V}}_t^l = \underline{V}_0^l + \sum_{s=t-n+1}^{t} \frac{n - (t-s)}{n} \alpha \, \widetilde{CF}_s^u .$$

Every summand $\frac{n-(t-s)}{n} \alpha \, \widetilde{CF}_s^u$ is normally distributed with expectation $\frac{n-(t-s)}{n} \alpha CF_0^u$ and variance $\left(\frac{n-(t-s)}{n}\alpha\right)^2 t$. Hence, the whole sum is again normally distributed. It has expectation

$$\underline{V}_0^l + \sum_{s=t-n+1}^{t} \frac{n - (t-s)}{n} \alpha CF_0^u = \underline{V}_0^l + \frac{n+1}{2} \alpha CF_0^u .$$

When evaluating the variance we have to take into account that the $\widetilde{CF}_s^u$ are correlated. We assume $t > n$ for simplicity and get

$$\operatorname{Var}\left[\sum_{s=t-n+1}^{t} \frac{n-(t-s)}{n}\, \alpha\, \widetilde{CF}_s^u\right] = \alpha^2 \operatorname{Var}\left[\sum_{s=t-n+1}^{t} \frac{n-(t-s)}{n} \sum_{r=1}^{s} \varepsilon_r\right]$$
$$= \alpha^2 \operatorname{Var}\left[\frac{1}{n}\sum_{s=1}^{t-n+1} \varepsilon_s + \frac{2}{n}\sum_{s=1}^{t-n+2} \varepsilon_s + \ldots + \frac{n}{n}\sum_{s=1}^{t} \varepsilon_s\right]$$
$$= \alpha^2 \operatorname{Var}\left[\sum_{s=1}^{t-n+1} \left\{\frac{1}{n} + \frac{2}{n} + \ldots + \frac{n}{n}\right\} \varepsilon_s + \left\{\frac{2}{n} + \ldots + \frac{n}{n}\right\} \varepsilon_{t-n+2} + \ldots + \left\{\frac{n}{n}\right\} \varepsilon_t\right]$$
$$= \alpha^2 \operatorname{Var}\left[\sum_{s=1}^{t-n+1} \frac{n(n+1)}{2n}\, \varepsilon_s + \frac{(n+2)(n+1-2)}{2n}\, \varepsilon_{t-n+2} \right.$$
$$\left. + \frac{(n+3)(n+1-3)}{2n}\, \varepsilon_{t-n+3} + \ldots + \frac{(n+n)(n+1-n)}{2n}\, \varepsilon_t\right] .$$

Since the ε are pairwise independent it follows that the variance is equal to

$$\alpha^2 \left[\sum_{s=1}^{t-n+1} \operatorname{Var}[\varepsilon_s] \frac{n^2(n+1)^2}{4n^2} + \frac{(n+2)^2(n+1-2)^2}{4n^2} \operatorname{Var}[\varepsilon_{t-n+2}] \right.$$
$$\left. + \frac{(n+3)^2(n+1-3)^2}{4n^2} \operatorname{Var}[\varepsilon_{t-n+3}] + \ldots + \frac{(n+n)^2(n+1-n)^2}{4n^2} \operatorname{Var}[\varepsilon_t]\right] .$$

Since the variance of all noise terms is 1, this can be simplified to

$$\alpha^2 \left[\frac{n^2(n+1)^2(t-n)}{4n^2} + \frac{(n+1)^2(n+1-1)^2}{4n^2} + \frac{(n+2)^2(n+1-2)^2}{4n^2} \right.$$
$$\left. + \frac{(n+3)^2(n+1-3)^2}{4n^2} + \ldots + \frac{(n+n)^2(n+1-n)^2}{4n^2}\right] .$$

Although this is a complicated sum it can nevertheless be evaluated and we get

$$\alpha^2 \left[\frac{n^2(n+1)^2(t-n)}{4n^2} + \frac{(n+1)(1+2n)\left(1+2n+2n^2\right)}{30n}\right]$$

and after simplification,

$$\alpha^2\, \frac{(1+n)(2+15tn(n+1)+8n-3n^2-7n^3)}{60n} .$$

This is the variance of the book value of the firm.

These evaluations show two interesting things:

– The expectation of the future book value does not depend on time. It stays constant at a level above the book value today.
– The variance does depend on time. With higher t the variance increases, the increase is linear with slope $\alpha \frac{(n+1)^2}{4}$.

Solution 2.19 The value of the firm that is financed by book value is given by

$$\begin{aligned}
V_0^l(\text{finan. book value}) &= V_0^u + \tau D_0 + \frac{nr_f - 1 + \left(1 + r_f\right)^{-n}}{nr_f} \tau\alpha\underline{l}V_0^u \\
&= \frac{100}{0.15} + 0.34 \times 500 \\
&\quad + \frac{4 \times 0.05 - 1 + (1 + 0.05)^{-4}}{4 \times 0.05} \times 0.34 \times 0.5 \times 0.7 \times \frac{100}{0.15} \\
&\approx 845.67\ .
\end{aligned}$$

The Miles-Ezzell formula for infinite lifetime (see Prob. 2.17 in Sect. 2.5.5.3) can be applied, but l_0 is not known (if l_0 were equal to $\underline{l}$ then book and market value would coincide). Hence,

$$\begin{aligned}
V_0^l(\text{finan. market value}) &= \frac{\mathrm{E}\left[\widetilde{CF}_t^u\right]}{(1 + k^{E,u})\left(1 - \frac{\tau r_f}{1+r_f} l_0\right) - 1} \\
&= \frac{\widetilde{CF}_t^u}{(1 + k^{E,u})\left(1 - \frac{\tau r_f}{1+r_f} \frac{D_0}{V_0^l(\text{finan. market value})}\right) - 1}\ .
\end{aligned}$$

This can be rearranged to

$$\begin{aligned}
V_0^l(\text{finan. market value}) &= \frac{\mathrm{E}\left[\widetilde{CF}_t^u\right]}{k^{E,u}} + \frac{\tau r_f}{1 + r_f} \frac{1 + k^{E,u}}{k^{E,u}} D_0 \\
&= \frac{100}{0.15} + \frac{0.34 \times 0.05}{1 + 0.05} \frac{1 + 0.15}{0.15} 500 \\
&\approx 728.73\ .
\end{aligned}$$

Obviously, the difference is very large.

Solution 2.20 First, if $n \to \infty$ and $T \to \infty$

$$\lim_{n\to\infty} \delta^n = \lim_{T\to\infty} \delta^T = 0$$

since $\delta < 1$. The same holds for γ and $\frac{1}{1+r_f}$. Now Thm. 2.23 reduces to

$$\begin{aligned}\lim_{n\to\infty} V_0^l &= (1 - 0\,(1 - \tau\,(1-0)))\,D_0 \\ &+ (1 - 0\,(1 - \tau\,(1-0)) - \tau\,(1-0))\,\frac{Div}{r_f(1-\tau)} \\ &+ \left(0 - 0 + \frac{0}{\frac{\gamma}{\delta}-1}\,\frac{k^{E,u}-g}{1+g}\right)\frac{V_0^u}{1-0}\end{aligned}$$

which was to be shown.

Solution 2.21 In a binomial model, the individual probabilities Q can be determined (see Fig. 2.3).

a) Using (2.14) this gives

$$\begin{aligned}V_0^l &= V_0^u + (\widetilde{D}_1(u)Q_1(u) + \widetilde{D}_1(d)Q_1(d))\frac{\tau r_f}{(1+r_f)^2} \\ &= 159.72 + (\widetilde{D}_1(u)\times 0.083 + \widetilde{D}_1(d)\times 0.917)\times 0.0281\ .\end{aligned}$$

b) Since

$$100 = \mathrm{E}\left[\widetilde{D}_1\right] = \frac{1}{2}\left(\widetilde{D}_1(d) + \widetilde{D}_1(u)\right)$$

and since debt cannot be negative, the highest value of V_0^l is achieved for

$$\widetilde{D}_1(d) = 200, \qquad \widetilde{D}_1(u) = 0\ .$$

This yields a firm value of 164.87 .

6.3 Personal Income Tax

Solution 3.1 Due to Thm. 3.4, we get

$$\begin{aligned}\frac{\mathrm{E}^Q\left[\widetilde{CF}_1^u\right]}{1+r_f(1-\tau)} &= \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}} \\ \frac{Q_1(u)\widetilde{CF}_1^u(u) + Q_1(d)\widetilde{CF}_1^u(d)}{1+r_f(1-\tau)} &= \frac{\mathrm{E}\left[\widetilde{CF}_1^u\right]}{1+k^{E,u}} \\ \frac{Q_1(u)\times 110 + Q_1(d)\times 90}{1.05} &= \frac{100}{1.2}\ ,\end{aligned}$$

and from this

$$Q_1(u) \approx -0.125, \qquad Q_1(d) \approx 1.125\ .$$

Any claim that pays one dollar after tax if up and nothing after tax if *down* must have a price of

$$V_0^u = \frac{Q_1(u)}{1 + r_f(1-\tau)} < 0$$

and this is an arbitrage opportunity. If $k^{E,u} = 15\%$ we get

$$Q_1(u) \approx 0.0652, \qquad Q_1(d) \approx 0.9348 .$$

To evaluate $Q_2(dd), \ldots$ we concentrate on $t = 1$. Analogously, we must have

$$\frac{Q_2(u|u) \times 132 + Q_2(d|u) \times 110}{1.05} = \frac{121}{1.15} ,$$

which gives

$$Q_2(u|u) \approx 0.0217, \quad Q_2(d|u) \approx 0.9783 .$$

Now by Bayes theorem

$$Q_2(uu) = Q_1(u) \times Q_2(u|u) \approx 0.0014$$
$$Q_2(ud) = Q_1(u) \times Q_2(u|d) \approx 0.0638$$

and analogously

$$Q_2(du) \approx 0.1016, \qquad Q_2(dd) \approx 0.8332 .$$

Solution 3.2 (3.7) implies

$$\widetilde{V}^l_{t+1} = \widetilde{V}^u_{t+1} + (1-\tau^D)\widetilde{A}_{t+1} + \frac{\mathrm{E}^Q_{t+1}\left[\tau^I r_f (1-\tau^D)\widetilde{A}_{t+1}\right]}{1 + r_f(1-\tau^I)} + \ldots + \frac{\mathrm{E}^Q_{t+1}\left[\tau^I r_f (1-\tau^D)\widetilde{A}_{T-1}\right]}{(1 + r_f(1-\tau^I))^{T-t-1}} .$$

Rule 4 (Iterated Expectations) gives

$$\mathrm{E}^Q_t\left[\widetilde{V}^l_{t+1} - \widetilde{V}^u_{t+1}\right] = \mathrm{E}^Q_t\left[(1-\tau^D)\widetilde{A}_{t+1}\right] + \frac{\mathrm{E}^Q_t\left[\tau^I r_f (1-\tau^D)\widetilde{A}_{t+1}\right]}{1 + r_f(1-\tau^I)} + \ldots + \frac{\mathrm{E}^Q_t\left[\tau^I r_f (1-\tau^D)\widetilde{A}_{T-1}\right]}{(1 + r_f(1-\tau^I))^{T-t-1}}$$

or

$$\frac{\mathrm{E}^Q_t\left[\widetilde{V}^l_{t+1} - \widetilde{V}^u_{t+1} - (1-\tau^D)\widetilde{A}_{t+1}\right]}{1 + r_f(1-\tau^I)} = \frac{\mathrm{E}^Q_t\left[\tau^I r_f (1-\tau^D)\widetilde{A}_{t+1}\right]}{(1 + r_f(1-\tau^I))^2} + \ldots + \frac{\mathrm{E}^Q_t\left[\tau^I r_f (1-\tau^D)\widetilde{A}_{T-1}\right]}{(1 + r_f(1-\tau^I))^{T-t}} .$$

Now adding $\frac{\mathrm{E}_t^Q\left[\tau^I r_f(1-\tau^D)\widetilde{A}_t\right]}{1+r_f(1-\tau^I)}$ results in

$$\frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l-\widetilde{V}_{t+1}^u\right]}{1+r_f(1-\tau^I)}+\frac{\mathrm{E}_t^Q\left[\tau^I r_f(1-\tau^D)\widetilde{A}_t-(1-\tau^D)\widetilde{A}_{t+1}\right]}{1+r_f(1-\tau^I)}=$$
$$\frac{\mathrm{E}_t^Q\left[\tau^I r_f(1-\tau^D)\widetilde{A}_t\right]}{1+r_f(1-\tau^I)}+\frac{\mathrm{E}_t^Q\left[\tau^I r_f(1-\tau^D)\widetilde{A}_{t+1}\right]}{(1+r_f(1-\tau^I))^2}+\ldots$$
$$+\frac{\mathrm{E}_t^Q\left[\tau^I r_f(1-\tau^D)\widetilde{A}_{T-1}\right]}{(1+r_f(1-\tau^I))^{T-t}}=\widetilde{V}_t^l-\widetilde{V}_t^u-(1-\tau^D)\widetilde{A}_t\ .$$

Reshuffling gives

$$\widetilde{V}_t^l-\widetilde{V}_t^u=\frac{\mathrm{E}_t^Q\left[\widetilde{V}_{t+1}^l-\widetilde{V}_{t+1}^u\right]}{1+r_f(1-\tau^I)}+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t-\frac{\mathrm{E}_t^Q\left[(1-\tau^D)\widetilde{A}_{t+1}\right]}{1+r_f(1-\tau^I)}\ .$$

Solution 3.3 We have

$$\begin{aligned}
\widetilde{V}_t^l &= \widetilde{V}_t^u+\left(1-\tau^D\right)\widetilde{A}_t+\sum_{s=t}^{T}\frac{\tau^I\left(1-\tau^D\right)r_f\,\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t+1}}\\
&= \widetilde{V}_t^u+\left(1-\tau^D\right)\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)r_f\widetilde{A}_t}{1+r_f\left(1-\tau^I\right)}+\sum_{s=t+1}^{T}\frac{\tau^I\left(1-\tau^D\right)r_f\,\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t+1}}\\
&= \widetilde{V}_t^u+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t+\sum_{s=t+1}^{T}\frac{\tau^I\left(1-\tau^D\right)r_f\,\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t+1}}\\
&= \widetilde{V}_t^u+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t}}\frac{(1-\tau^I)r_f}{1+r_f(1-\tau^I)}\\
&= \widetilde{V}_t^u+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t}}\left(1-\frac{1}{1+r_f(1-\tau^I)}\right)\\
&= \widetilde{V}_t^u+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t}}\\
&\quad -\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t+1}}\\
&= \widetilde{V}_t^u+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{(1+r_f\left(1-\tau^I\right))^{s-t}}
\end{aligned}$$

$$-\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+2}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_{s-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}}$$

using $\widetilde{A}_T=0$. Rearranging gives

$$\widetilde{V}_t^l=\widetilde{V}_t^u+\frac{(1+r_f)(1-\tau^D)}{1+r_f(1-\tau^I)}\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}}$$
$$-\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_{s-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}}+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\frac{\widetilde{A}_t}{1+r_f(1-\tau^I)}$$

and this is equivalent to

$$\widetilde{V}_t^l=\widetilde{V}_t^u+\frac{1-\tau^D}{1-\tau^I}\widetilde{A}_t+\frac{\tau^I\left(1-\tau^D\right)}{1-\tau^I}\sum_{s=t+1}^{T}\frac{\mathrm{E}_t^Q\left[\widetilde{A}_s-\widetilde{A}_{s-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{s-t}}\,.$$

Solution 3.4 This follows from

$$\begin{aligned}k^{\text{post-tax}}&=\frac{\mathrm{E}_t\left[\widetilde{GCF}_{t+1}+\widetilde{V}_{t+1}^u-\tau(\widetilde{GCF}_{t+1}+\widetilde{V}_{t+1}^u-\widetilde{V}_t^u)\right]}{\widetilde{V}_t^u}\\&=(1-\tau)\frac{\mathrm{E}_t\left[\widetilde{GCF}_{t+1}+\widetilde{V}_{t+1}^u\right]}{\widetilde{V}_t^u}\\&=(1-\tau)k^{\text{pre-tax}}\,.\end{aligned}$$

Solution 3.5 We have

$$\begin{aligned}
\widetilde{V}_t^u &= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t^Q\left[\widetilde{GCF}_s(1-\tau)\right]}{(1+r_f(1-\tau))^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t^Q\left[\widetilde{GCF}_s\right]}{\left(1+r_f\right)^{s-t}} \frac{(1-\tau)\left(1+r_f\right)^{s-t}}{(1+r_f(1-\tau))^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{\mathrm{E}_t\left[\widetilde{GCF}_s\right]}{(1+k^{\text{pre-tax}})^{s-t}} \frac{(1-\tau)\left(1+r_f\right)^{s-t}}{(1+r_f(1-\tau))^{s-t}} \\
&= \sum_{s=t+1}^{\infty} \frac{\widetilde{GCF}_t}{(1+k^{\text{pre-tax}})^{s-t}} \frac{(1-\tau)\left(1+r_f\right)^{s-t}}{(1+r_f(1-\tau))^{s-t}} \\
&= \frac{(1-\tau)\widetilde{GCF}_t}{\frac{(1+k^{\text{pre-tax}})(1+r_f(1-\tau))}{1+r_f} - 1}
\end{aligned}$$

and the denominator $(1+k^{\text{pre-tax}})\frac{1+r_f(1-\tau)}{1+r_f}-1$is obviously different from $k^{\text{pre-tax}}(1-\tau)$.

Solution 3.6

a) As in Sect. 2.7.1 we get

$$\begin{aligned}
\frac{\mathrm{E}^Q\left[\widetilde{GCF}_1\right]}{1+r_f} &= \frac{\mathrm{E}\left[\widetilde{GCF}_1\right]}{1+k} \\
\frac{Q_1(u)\widetilde{GCF}_1(u)+Q_1(d)\widetilde{GCF}_1(d)}{1+r_f} &= \frac{\mathrm{E}\left[\widetilde{GCF}_1\right]}{1+k} \\
\frac{Q_1(u)\times 110+Q_1(d)\times 90}{1.05} &= \frac{100}{1.15}\ ,
\end{aligned}$$

and hence

$$Q_1(u)\approx 0.0652, \qquad Q_1(d)\approx 0.93478\ .$$

b) If we look at the second company, the gross cash flows are different and we get

$$\begin{aligned}
\frac{\mathrm{E}^Q\left[\widetilde{GCF}'_1\right]}{1+r_f} &= \frac{\mathrm{E}\left[\widetilde{GCF}'_1\right]}{1+k'} \\
\frac{Q_1(u)\widetilde{GCF}'_1(u)+Q_1(d)\widetilde{GCF}'_1(d)}{1+r_f} &= \frac{\mathrm{E}\left[\widetilde{GCF}'_1\right]}{1+k'} \\
\frac{0.0652\times 120+0.93478\times 80}{1.05} &\approx \frac{100}{1+k'}\ ,
\end{aligned}$$

and hence

$$k'\approx 27.105\%\ .$$

c) This is the same calculation except that we have to add taxes:

$$\begin{aligned}
\frac{\mathrm{E}^Q\left[(1-\tau)\widetilde{GCF}_1\right]}{1+r_f(1-\tau)} &= \frac{\mathrm{E}\left[(1-\tau)\widetilde{GCF}_1\right]}{1+k(1-\tau)} \\
\frac{Q_1(u)(1-\tau)\widetilde{GCF}_1(u)+Q_1(d)(1-\tau)\widetilde{GCF}_1(d)}{1+r_f(1-\tau)} &= \frac{\mathrm{E}\left[(1-\tau)\widetilde{GCF}_1\right]}{1+k(1-\tau)} \\
\frac{Q_1(u)(1-\tau)110+Q_1(d)(1-\tau)90}{1+0.05(1-\tau)} &= \frac{100(1-\tau)}{1+0.15(1-\tau)}\ ,
\end{aligned}$$

and hence

$$Q_1(u) \approx 2.8333\,\frac{(0.176471+\tau)}{7.6667-\tau}, \qquad Q_1(d) = 1-Q_1(u)\ .$$

d) The calculation is as above,

$$\begin{aligned}
\frac{\mathrm{E}^Q\left[(1-\tau)\widetilde{GCF}_1\right]}{1+r_f(1-\tau)} &= \frac{\mathrm{E}\left[(1-\tau)\widetilde{GCF}_1\right]}{1+k'(1-\tau)} \\
\frac{Q_1(u)(1-\tau)\widetilde{GCF}_1(u)+Q_1(d)(1-\tau)\widetilde{GCF}_1(d)}{1+r_f(1-\tau)} &= \frac{\mathrm{E}\left[(1-\tau)\widetilde{GCF}_1\right]}{1+k'(1-\tau)} \\
\frac{Q_1(u)(1-\tau)120+Q_1(d)(1-\tau)80}{1+0.05(1-\tau)} &= \frac{100(1-\tau)}{1+15.5\%(1-\tau)}\ ,
\end{aligned}$$

and hence

$$Q_1(u) \approx 1.1667\,\frac{(1.85714+\tau)}{7.6667-\tau}, \qquad Q_1(d) = 1-Q_1(u)\ .$$

This is different from the result in 6.3.

Solution 3.7 From (3.7) it follows that the market value is maximized if the retentions are as large as possible. Since necessarily

$$\widetilde{A}_3 = 0\ ,$$

we only look at $\widetilde{A}_1$ and $\widetilde{A}_2$. We will determine the highest possible retention and evaluate the corresponding market value of the firm.

If the company maintains its highest possible retention, the cash flows to the shareholders are zero. From this

$$\widetilde{A}_1 := \frac{1}{1-\tau^D}\widetilde{CF}_1^u, \qquad \widetilde{A}_2 := \left(1+r_f\right)\widetilde{A}_1 + \frac{1}{1-\tau^D}\widetilde{CF}_2^u\ .$$

We put this into (3.7) and get, using theorem 3.4,

$$\begin{aligned}
V_0^l &= V_0^u + \frac{\tau^I r_f (1-\tau^D)\,\mathrm{E}^Q\left[\widetilde{A}_1\right]}{(1+r_f(1-\tau^I))^2} + \frac{\tau^I r_f (1-\tau^D)\,\mathrm{E}^Q\left[\widetilde{A}_2\right]}{(1+r_f(1-\tau^I))^3} \\
&= V_0^u + \frac{\tau^I r_f (1-\tau^D)\,\mathrm{E}^Q\left[\frac{\widetilde{CF}_1^u}{1-\tau^D}\right]}{(1+r_f(1-\tau^I))^2} + \frac{\tau^I r_f (1-\tau^D)\,\mathrm{E}^Q\left[\frac{(1+r_f)\,\widetilde{CF}_1^u+\widetilde{CF}_2^u}{1-\tau^D}\right]}{(1+r_f(1-\tau^I))^3} \\
&= V_0^u + \frac{\tau^I r_f\,\mathrm{E}^Q\left[\widetilde{CF}_1^u\right]}{(1+r_f(1-\tau^I))^2} + \frac{\tau^I r_f (1+r_f)\,\mathrm{E}^Q\left[\widetilde{CF}_1^u\right]}{(1+r_f(1-\tau^I))^3} + \frac{\tau^I r_f\,\mathrm{E}^Q\left[\widetilde{CF}_2^u\right]}{(1+r_f(1-\tau^I))^3} \\
&= V_0^u + \frac{\tau^I r_f\,\mathrm{E}\left[\widetilde{CF}_1^u\right]}{(1+k^{E,u})(1+r_f(1-\tau^I))} + \frac{\tau^I r_f (1+r_f)\,\mathrm{E}\left[\widetilde{CF}_1^u\right]}{(1+k^{E,u})(1+r_f(1-\tau^I))^2} + \\
&\qquad + \frac{\tau^I r_f\,\mathrm{E}\left[\widetilde{CF}_2^u\right]}{(1+k^{E,u})^2(1+r_f(1-\tau^I))} \\
&\approx 249.692 + \frac{0.5\times 0.1\times 100}{(1+0.15)\times(1+0.1\times(1-0.5))} \\
&\qquad + \frac{0.5\times 0.1\times(1+0.1)\times 100}{(1+0.15)\times(1+0.1\times(1-0.5))^2} + \frac{0.5\times 0.1\times 110}{(1+0.15)^2\times(1+0.1\times(1-0.5))} \\
&\approx 260.06\ .
\end{aligned}$$

Solution 3.8 With retention in risk-free assets (3.3) reads

$$\mathrm{E}_{t-1}^Q\left[\widetilde{CF}_t^l - \widetilde{CF}_t^u\right] = \left(1-\tau^I\right)\left(1+r_f\right)\widetilde{A}_{t-1} - \left(1-\tau^I\right)\mathrm{E}_{t-1}^Q\left[\widetilde{A}_t\right]\ .$$

Following Eq. (3.7) we get

$$\begin{aligned}
\widetilde{V}_t^l = \widetilde{V}_t^u &+ \frac{(1-\tau^I)\,\mathrm{E}_t^Q\left[\left(1+r_f\right)\widetilde{A}_t - \widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)} + \ldots \\
&+ \frac{(1-\tau^I)\,\mathrm{E}_t^Q\left[\left(1+r_f\right)\widetilde{A}_{T-2} - \widetilde{A}_{T-1}\right]}{1+r_f\left(1-\tau^I\right)^{T-t-1}} \\
&+ \frac{(1-\tau^I)\,\mathrm{E}_t^Q\left[\left(1+r_f\right)\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}}\ .
\end{aligned}$$

After some minimal reshuffling the following results

$$\widetilde{V}_t^l = \widetilde{V}_t^u + \frac{\left(1-\tau^I\right)\left(1+r_f\right)\widetilde{A}_t}{1+r_f\left(1-\tau^I\right)}$$

$$+\frac{\mathrm{E}_t^Q\left[\frac{(1+r_f)(1-\tau^I)}{1+r_f(1-\tau^I)}\widetilde{A}_{t+1}-\left(1-\tau^I\right)\widetilde{A}_{t+1}\right]}{1+r_f\left(1-\tau^I\right)}+\ldots$$
$$+\frac{\mathrm{E}_t^Q\left[\frac{(1+r_f)(1-\tau^I)}{1+r_f(1-\tau^I)}\widetilde{A}_{T-1}-\left(1-\tau^I\right)\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}}\,.$$

This brings us to the conclusion,

$$\widetilde{V}_t^l=\widetilde{V}_t^u+\left(1-\tau^I\right)\widetilde{A}_t+\frac{\tau^I\left(1-\tau^I\right)r_f\,\mathrm{E}_t^Q\left[\widetilde{A}_t\right]}{1+r_f\left(1-\tau^I\right)}+\ldots$$
$$+\frac{\tau^I\left(1-\tau^I\right)r_f\,\mathrm{E}_t^Q\left[\widetilde{A}_{T-1}\right]}{\left(1+r_f\left(1-\tau^I\right)\right)^{T-t}}\,.$$

In case of autonomous retention, with an perpetual firm this simply yields

$$\widetilde{V}_t^l=\widetilde{V}_t^u+A\,.$$

6.4 Corporate and Personal Income Tax

Solution 4.1 We get

$$\begin{aligned}\widetilde{V}_t^l&=\widetilde{V}_t^u+\frac{\left(1-\tau^D\right)\left(1-\tau^C\right)}{1-\tau^I}A+\tau^C D\\&=500+\frac{(1-0.5)\times(1-0.5)}{1-0.5}10+0.5\times 100\\&=555\,.\end{aligned}$$

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