Lecture: Business Values

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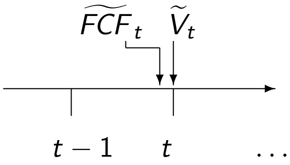
Discounted Cash Flow, Section 1.3

,

Outline

Trade and payment dates 1



The time structure of the model:

1. Buy in $t-1 \implies$ pay price $\widetilde{V}_{t-1}$.
2. Hold until $t \implies$ receive dividend $\widetilde{FCF}_t$.
3. Sell (and buy again) in $t \implies$ receive (and pay) price $\widetilde{V}_t$.

Notice that the investor receives **dividend** $\widetilde{FCF}_t$ **shortly before** t (the selling date).

What should happen if the future were **certain**?

In a certain world that is **free of arbitrage** we must have

$$V_t = \frac{FCF_{t+1} + V_{t+1}}{1 + r_f} \, .$$

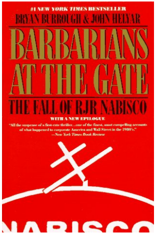
Otherwise, if for example

$$FCF_{t+1} + V_{t+1} > (1 + r_f)V_t$$

$\Longrightarrow$ take loan, buy share in t, wait until $t + 1$, get dividend and sell share

$\Longrightarrow$ get infinitely rich without any cost.

A risky world



The account of the largest takeover in Wall Street history...

In a risky world we have

$$\widetilde{FCF}_{t+1} = \begin{cases} 110 & \text{up,} \\ 90 & \text{down.} \end{cases}$$

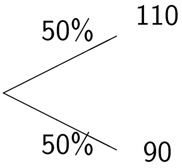
What is absence of arbitrage now?

$$V_t \neq \frac{\mathrm{E}\left[\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1}\right]}{1 + r_f}.$$

Certainty equivalent	Risk premium
$\dfrac{\mathrm{E}\left[\widetilde{FCF}_{t+1}+\widetilde{V}_{t+1}\mid\mathcal{F}_t\right]-\text{risk adjustment}}{1+r_f}$	$\dfrac{\mathrm{E}\left[\widetilde{FCF}_{t+1}+\widetilde{V}_{t+1}\mid\mathcal{F}_t\right]}{1+r_f+\text{risk premium}}$

Risk-neutral probability
$\dfrac{\mathrm{E}_Q\left[\widetilde{FCF}_{t+1}+\widetilde{V}_{t+1}\mid\mathcal{F}_t\right]}{1+r_f}$

We are going to illustrate the three roads by using the following example:



and $r_f = 0.05$. Because the cash flows have expectation

$$0.5 \times 110 + 0.5 \times 90 = 100,$$

the value V_0 will be less than $\frac{100}{1.05} \approx 95.24$.

Daniel Bernoulli, founder of utility theory

If we assume a utility function

$$u(x) = \sqrt{x},$$

then the certainty equivalent (CEQ) and hence the price of the asset is given by

$$u(\mathrm{CEQ}) = \mathrm{E}[u(x)]$$
$$\sqrt{CEQ} = 0.5 \times \sqrt{110} + 0.5 \times \sqrt{90}$$
$$CEQ \approx 99.75$$
$$V_0 = \frac{CEQ}{1 + r_f} \approx \frac{99.75}{1.05} = 95.$$

Valuation with a risk premium uses another idea: we modify the denominator

$$V_0 = \frac{\mathrm{E}\left[\widetilde{FCF}_1 + \widetilde{V}_1\right]}{1 + r_f + z}.$$

Using the numbers of the example we get

$$V_0 = \frac{100}{1 + 0.05 + z} \qquad \Longrightarrow \qquad z \approx 0.00263$$

$$V_0 = \frac{100}{1 + 0.05 + 0.00263} = 95.$$

With 'risk-neutral probabilities' the probabilities $p_u = 0.5$ and $p_d = 0.5$ are modified. The job is: find q_u and q_d such that

$$V_0 \overset{!}{=} \frac{\mathrm{E}_Q\left[\widetilde{FCF}_1 + \widetilde{V}_1\right]}{1 + r_f}$$

holds.

Answer:

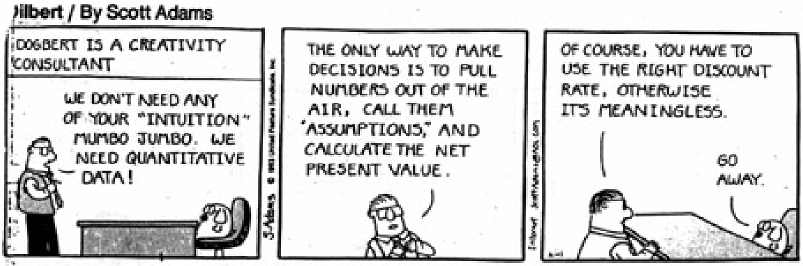
$$95 = \frac{q_u \times 110 + q_d \times 90}{1 + 0.05}$$

$$95 = \frac{q_u \times 110 + (1 - q_u) \times 90}{1 + 0.05}$$

$$\implies \quad q_u = 0.4875, \quad q_d = 0.5125.$$

We turn to this approach in more detail!

Cost of capital is definitely a key concept. But: how is it **precisely defined**?



Dilbert is a creative consultant:
We don't need any of your 'intuition' Mumbo-Jumbo, we need quantitative data!

The only way to make decisions is to pull numbers out of the air, call them 'assumptions' and calculate the net present value.

Of course, you need to use the right discount rate, otherwise its meaningless. – Go away.

There are several alternative definitions of cost of capital in a multi–period context:

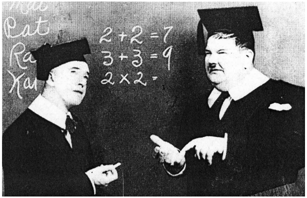
- cost of capital $=_{\text{Def}}$ 'Yields'
- cost of capital $=_{\text{Def}}$ 'Discount rates'
- cost of capital $=_{\text{Def}}$ 'Expected returns'
- cost of capital $=_{\text{Def}}$ 'Opportunity costs'

Are all definitions logically identical? We will show later: **not necessarily**!

Which definition is **useful for our purpose**? The following definition turns out to be appropriate

Definition 1.1 (cost of capital): *The cost of capital of a firm is the conditional expected return*

$$\tilde{k}_t =_{\text{Def}} \frac{\mathsf{E}\left[\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1} | \mathcal{F}_t\right]}{\widetilde{V}_t} - 1.$$



This appropriate definition has a disadvantage: k_t could be uncertain. And **you cannot discount with uncertain cost of capital!**

Another (big) assumption is necessary: from now on this cost of capital **should be certain**.

Theorem 1.1 (market value): *When cost of capital is deterministic, then*

$$\widetilde{V}_t = \sum_{s=t+1}^{T} \frac{\mathrm{E}\left[\widetilde{FCF}_s | \mathcal{F}_t\right]}{(1+k_t)\cdots(1+k_{s-1})}.$$

Notice that the lefthand side and the righthand side as well can be uncertain for $t > 0$.

Proof of Theorem 1.1

Start with a reformulation of Definition 1.1,

$$\widetilde{V}_t = \frac{\mathrm{E}\left[\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1} | \mathcal{F}_t\right]}{1 + k_t}.$$

Use the corresponding relation for $\widetilde{V}_{t+1}$ and plug in, using rule 2 (linearity) because cost of capital are deterministic

$$\begin{aligned}
\widetilde{V}_t &= \frac{\mathrm{E}\left[\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1} | \mathcal{F}_t\right]}{1 + k_t} \\
&= \frac{\mathrm{E}\left[\widetilde{FCF}_{t+1} + \frac{\mathrm{E}\left[\widetilde{FCF}_{t+2} + \widetilde{V}_{t+2} | \mathcal{F}_{t+1}\right]}{1+k_{t+1}} | \mathcal{F}_t\right]}{1 + k_t} \\
&= \frac{\mathrm{E}\left[\widetilde{FCF}_{t+1} | \mathcal{F}_t\right]}{1+k_t} + \frac{\mathrm{E}\left[\mathrm{E}\left[\widetilde{FCF}_{t+2} | \mathcal{F}_{t+1}\right] | \mathcal{F}_t\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathrm{E}\left[\mathrm{E}\left[\widetilde{V}_{t+2} | \mathcal{F}_{t+1}\right] | \mathcal{F}_t\right]}{(1+k_t)(1+k_{t+1})}
\end{aligned}$$

Rule 4 (iterated expectation) gives

$$\widetilde{V}_t = \frac{\mathsf{E}\left[\widetilde{FCF}_{t+1}|\mathcal{F}_t\right]}{1+k_t} + \frac{\mathsf{E}\left[\widetilde{FCF}_{t+2}|\mathcal{F}_t\right]}{(1+k_t)(1+k_{t+1})} + \frac{\mathsf{E}\left[\widetilde{V}_{t+2}|\mathcal{F}_t\right]}{(1+k_t)(1+k_{t+1})}.$$

Continue until T to get

$$\begin{aligned}\widetilde{V}_t =& \frac{\mathsf{E}\left[\widetilde{FCF}_{t+1}|\mathcal{F}_t\right]}{1+k_t} + \frac{\mathsf{E}\left[\widetilde{FCF}_{t+2}|\mathcal{F}_t\right]}{(1+k_t)(1+k_{t+1})} + \dots \\ &+ \frac{\mathsf{E}\left[\widetilde{FCF}_T|\mathcal{F}_t\right]}{(1+k_t)\cdots(1+k_{T-1})} + \frac{\mathsf{E}\left[\widetilde{V}_T|\mathcal{F}_t\right]}{(1+k_t)\cdots(1+k_{T-1})}.\end{aligned}$$

Last term vanishes by transversality. QED.

Other attempts to modify definition 1.1 fail to produce reasonable results: aim is to have

$$V_0 = \frac{\mathrm{E}\left[\widetilde{FCF}_1\right]}{1+k_0} + \frac{\mathrm{E}\left[\widetilde{FCF}_2\right]}{(1+k_0)(1+k_1)} + \dots$$

as well as for $t = 1$

$$\widetilde{V}_1 = \frac{\mathrm{E}\left[\widetilde{FCF}_2|\mathcal{F}_1\right]}{1+k_1} + \frac{\mathrm{E}\left[\widetilde{FCF}_3|\mathcal{F}_1\right]}{(1+k_1)(1+k_2)} + \dots$$

with the same cost of capital k_1 in the denominator!

For example:

$$k =_{\text{Def?}} \frac{\mathrm{E}\left[\widetilde{V}_{t+1} + \widetilde{FCF}_{t+1}\right]}{\mathrm{E}[\widetilde{V}_t]} - 1$$

cannot be rearranged to $\widetilde{V}_t = \ldots$ (and is not an expected return!).

The same applies to

$$k =_{\text{Def?}} \mathrm{E}\left[\frac{\widetilde{V}_{t+1} + \widetilde{FCF}_{t+1}}{\widetilde{V}_t} - 1\right]$$

Rapp (zfbf, 2006) and Laitenberger (zfb, 2006) have analyzed that problem (in German).

Let us turn back to risk-neutral probability Q: does Q always exist? This is not a trivial question: the numbers q_u and q_d must be between 0 and 1! For example, $(q_u, q_d) = (-1, 2)$ would not be considered as 'probabilities'.

Theorem 1.2 (fundamental theorem): *If the markets are free of arbitrage, there is a probability Q such that for* **all** *claims*

$$\widetilde{V}_t = \frac{\mathrm{E}_Q\left[\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1} | \mathcal{F}_t\right]}{1 + r_f}.$$

Holds for any claims!

What about a proof? Forget it.[1]

How to get Q for valuation of firms? No idea.

So why is this helpful? We will see (much) later.

Is there at least an interpretation of Q? Yes!

[1] If you cannot resist: see further literature. . .

Why do we call Q a risk-neutral probability?

Look at the following:

$$1 + r_f = \frac{\mathrm{E}_Q\left[\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1} | \mathcal{F}_t\right]}{\widetilde{V}_t} \qquad \text{fundamental theorem}$$

$$1 + r_f = \mathrm{E}_Q\left[\left.\frac{\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t}\right| \mathcal{F}_t\right] \qquad \text{rule 5}$$

$$r_f = \mathrm{E}_Q\left[\left.\frac{\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t}\right| \mathcal{F}_t\right] - \mathrm{E}_Q\left[1 | \mathcal{F}_t\right] \qquad \text{rule 3}$$

$$r_f = \mathrm{E}_Q\left[\left.\underbrace{\frac{\widetilde{FCF}_{t+1} + \widetilde{V}_{t+1}}{\widetilde{V}_t} - 1}_{\text{return on holding a share}}\right| \mathcal{F}_t\right] \qquad \text{rule 2.}$$

Intuition of Q

The last equation

$$r_f = \mathrm{E}_Q[\text{return}|\mathcal{F}_t]$$

simply says: if we change our probabilities to Q, any security has expected return r_f.

Or: **the world is risk-neutral under Q.**

Uniqueness of Q

Is Q unique? Or can the value of the company depend on Q?

If the cash flows of the firm can be duplicated by traded assets **(market is complete)** then any Q will lead to the same value.

Proof? Forget this as well...

From now on two assumptions will always hold:

Assumption 1.1: *The markets are free of arbitrage.*

Assumption 1.2: *The cash flows of the firm can be duplicated by traded assets.*

$\Longrightarrow$ The risk-neutral probability Q exists and is (in some sense) unique.

Costs of capital are conditional expected returns.

Costs of capital must be deterministic.

If markets are arbitrage free a 'risk-neutral probability measure' Q exists.

When using this probability Q the world is risk-neutral.