

# **DCF: Firm income tax**

## **Conclusions from martingale-like cash flows**

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What are the implications of martingale-like cash flows?

Two important theorems:

deterministic d-p-ratio The dividend–price ratio is deterministic and not random.

cost of capital = discount rate The cost of capital of the unlevered firm can be used as a discount rate.



**Theorem (Williams, Gordon-Shapiro):** *Assume costs of capital are deterministic and cash flows are martingale-like. If  $k^{E,u} > g$ , then*

$$\tilde{V}_t^u = \frac{1+g}{k^{E,u}-g} \tilde{CF}_t^u$$

*holds.*

(Our first multiple!)



First notice that ( $s > t$ )

$$\begin{aligned} E_t \left[ \widetilde{CF}_s^u \right] &= E_t \left[ E_{s-1} \left[ \widetilde{CF}_s^u \right] \right] && \text{by rule 4} \\ &= E_t \left[ (1 + g) \widetilde{CF}_{s-1}^u \right] && \text{by assumption} \\ &= (1 + g) E_t \left[ \widetilde{CF}_{s-1}^u \right] && \text{by rule 2} \\ &= (1 + g) \cdots (1 + g) E_t \left[ \widetilde{CF}_t^u \right] && \text{continued} \\ &= (1 + g)^{s-t} \widetilde{CF}_t^u && \text{by rule 5.} \end{aligned}$$



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$$\tilde{V}_t^u = \sum_{s=t+1}^T \frac{E_t [\tilde{CF}_s^u]}{(1 + k^{E,u})^{s-t}}$$

from valuation theorem

$$= \underbrace{\sum_{s=t+1}^T \frac{(1+g)^{s-t}}{(1+k^{E,u})^{s-t}}}_{:= (1+g)/(k^{E,u}-g)} \tilde{CF}_t^u$$

see slide above

$$= \frac{1+g}{k^{E,u}-g} \tilde{CF}_t^u$$

QED



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see slide above

$$= \frac{1+g}{k^{E,u}-g} \tilde{CF}_t^u$$

QED



if  $a < 1$ <sup>1</sup>

$$S = a + a^2 + a^3 + \dots$$

$$aS = \quad a^2 + a^3 + \dots$$

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$$S - aS = a \quad \implies \quad S = \frac{a}{1 - a}$$

and now put  $a = \frac{1+g}{1+k^{E,u}}$  gives  $\frac{\frac{1+g}{1+k^{E,u}}}{1 - \frac{1+g}{1+k^{E,u}}} = \frac{1+g}{k^{E,u} - g}$

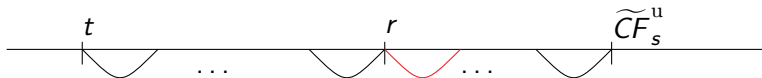
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<sup>1</sup>which is here the same as  $g < k^{E,u}$ .

We want to look at discount rates now. First let us precisely define them.

Notice that discount rates will depend

- on the cash flow we want to value ( $\widetilde{CF}_s^u$ ),
- on the point in time where we determine this value (index  $t$ ) and
- on the actual time period (index  $r$ ) we are discounting.



We use the (clumsy) notation  $\kappa_r^{t \rightarrow s}$  for discounting from  $r + 1$  to  $r$ .



**Definition discount rates:** Real numbers are called discount rates of the cash flow  $\widetilde{CF}_t^u$  if they satisfy

$$\underbrace{\frac{E_t^Q [\widetilde{CF}_s^u]}{(1+r_f)^{s-t}}}_{\text{value (fundam. theorem)}} = \underbrace{\frac{E_t [\widetilde{CF}_s^u]}{(1+\kappa_t^{t \rightarrow s}) \cdots (1+\kappa_{s-1}^{t \rightarrow s})}}_{\text{computational rule}}.$$

Interpretation of rhs: the way we apply discount rates.

Interpretation of lhs: value, follows from fundamental theorem.





Now, finally, our second implication from martingale-like cash flows.

**Theorem: Equivalence of valuation concepts** *If costs of capital are deterministic and cash flows are martingale-like, then*

$$\frac{E_t^Q \left[ \widetilde{CF}_s^u \right]}{(1 + r_f)^{s-t}} = \frac{E_t \left[ \widetilde{CF}_s^u \right]}{(1 + k^{E,u})^{s-t}}$$

or: costs of capital are discount rates!



Notice that **sums** are equal

$$\underbrace{\sum_{s=t+1}^T \frac{E_t^Q [\widetilde{CF}_s^u]}{(1+r_f)^{s-t}}}_{\text{fundamental theorem}} = \widetilde{V}_t^u = \underbrace{\sum_{s=t+1}^T \frac{E_t [\widetilde{CF}_s^u]}{(1+k^{E,u})^{s-t}}}_{\text{valuation theorem}}.$$

$$4 + 6 = 10 = 3 + 7$$

The theorem tells us that **summands** are equal as well

$$\frac{E_t^Q [\widetilde{CF}_s^u]}{(1+r_f)^{s-t}} = \frac{E_t [\widetilde{CF}_s^u]}{(1+k^{E,u})^{s-t}}.$$

$$4 \neq 3 \quad \text{and} \quad 6 \neq 7$$

This result is not trivial at all!





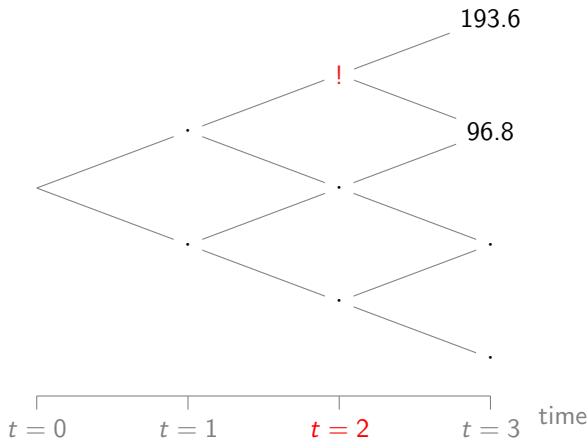
We skip the proof!



# Determining $Q$

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Let  $r_f = 10\%$ . Another additional result is the evaluation of the risk-neutral probability  $Q$ .



Fokus on !.

We want to value the future cash flow resulting from this node. There are two ways and both are equivalent (cost of capital is a discount rate)

$$\frac{Q(u) \times 193.6 + Q(d) \times 96.8}{1 + 10\%} = \frac{\frac{1}{2}193.6 + \frac{1}{2}96.8}{1 + 20\%} = 121.$$

Since  $Q(u) + Q(d) = 1$  this results in



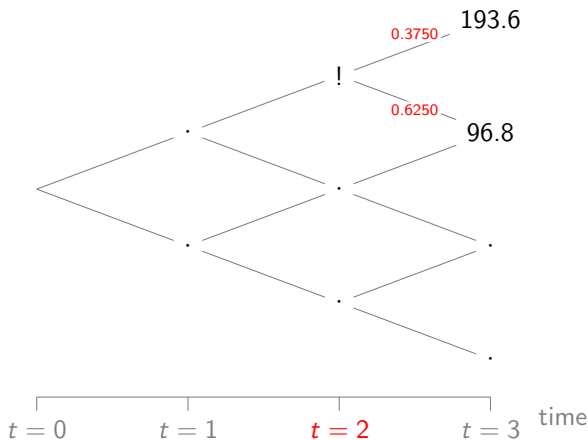


Figure:  $Q$  probabilities



Look at all nodes at  $t = 2$ . Then

$$\frac{E_2^Q [\widetilde{CF}_3^u]}{1 + r_f} = \frac{E_2 [\widetilde{CF}_3^u]}{1 + k^{E,u}}.$$

Then at node  $\omega^2$  this equation translates to

$$\begin{aligned} \frac{E_2^Q [\widetilde{CF}_3^u]}{1 + r_f} &= \frac{Q_3(u|\omega) \widetilde{CF}_3^u(u|\omega) + Q_3(d|\omega) \widetilde{CF}_3^u(d|\omega)}{1 + r_f} \\ &= \frac{P_3(u|\omega) \widetilde{CF}_3^u(u|\omega) + P_3(d|\omega) \widetilde{CF}_3^u(d|\omega)}{1 + k^{E,u}} = \frac{E_2 [\widetilde{CF}_3^u]}{1 + k^{E,u}}. \end{aligned}$$

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<sup>2</sup>We considered  $\omega = uu$  in last slide.

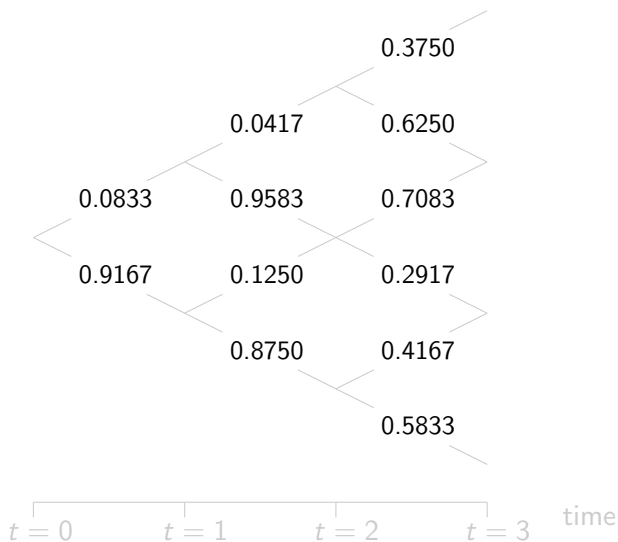


Also, the conditional  $Q$ -probabilities add to one:

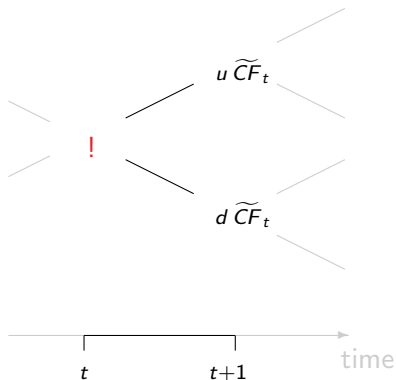
$$Q_3(u|\omega) + Q_3(d|\omega) = 1 .$$

This is a  $2 \times 2$ -system that can be solved for every  $\omega$ !





Fix a node again and apply the theorem relating the discount rate to the cost of capital.



Then

$$\frac{E_t^Q \left[ \widetilde{CF}_{t+1}^u \right]}{1 + r_f} = \frac{E_t \left[ \widetilde{CF}_{t+1}^u \right]}{1 + k^{E,u}}$$

$$\frac{Q(u) \times u \times \widetilde{CF}_t^u + Q(d) \times d \times \widetilde{CF}_t^u}{1 + r_f} = \frac{P(u) \times u \times \widetilde{CF}_t^u + P(d) \times d \times \widetilde{CF}_t^u}{1 + k^{E,u}}$$

$\widetilde{CF}_t^u$  cancels! We end up with

$$\frac{Q(u) \times u + Q(d) \times d}{1 + r_f} = \frac{\frac{1}{2}u + \frac{1}{2}d}{1 + k^{E,u}}$$

$$Q(u) + Q(d) = 1$$

Again 2x2 equation!



$Q$  can be determined:

$$Q_{t+1}(u|\omega) = \frac{\frac{1+r_f}{1+k^{E,u}} - d}{u - d}, \quad Q_{t+1}(d|\omega) = \frac{u - \frac{1+r_f}{1+k^{E,u}}}{u - d}$$

regardless of  $t$  and  $\omega$ .

*Remark:* The factors  $u$  and  $d$  cannot be chosen arbitrarily in the infinite case if the cost of capital  $k^{E,u}$  is given, because

$$d < \frac{1+r_f}{1+k^{E,u}} < u$$

must hold in order to ensure positive  $Q$ -probabilities.

With  $k^{E,u} = 20\%$  the value of the unlevered firm is

$$V_0^u = \frac{E[\widetilde{CF}_1^u]}{k^{E,u}} = 500.$$



The costs of capital of unlevered firm are discount rates.

The dividend–price ratio (a multiple) is deterministic.

We can determine  $Q$  in both examples.

