DCF: Basic Concepts
Information and Conditional Expectation

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Traditional approach to handling uncertainty 2

We have in mind an image of uncertainty in which the "fog becomes denser and denser" as we look further into the future.

Uncertainty is usually modeled via different **states of nature** ω with corresponding cash flows $\widetilde{CF}_t(\omega)$.

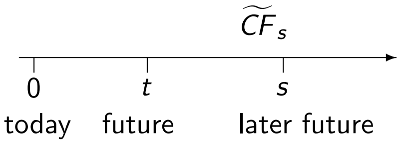
But: Typically, particular states of nature play no role in firms' valuation equations; instead, one **uses expectations** $\mathrm{E}\left[\widetilde{CF}_t\right]$ of cash flows.

Information: Remember the point-in-time principle 3

Today is risk-free, the future is random.

Now: **we always stay at time 0!**

And think about the future



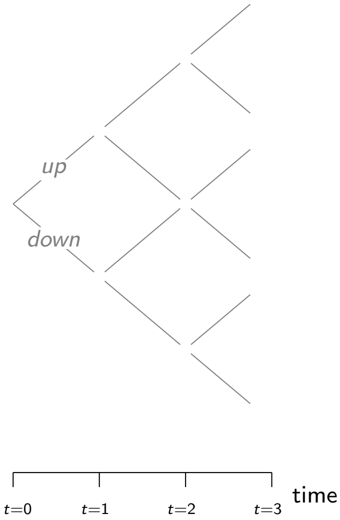
We need a mathematical tool

A.N. Kolmogorov (1903–1987), developed modern probability theory

We know what an expectation is: the value we expect today.

In DCF we need more—the value we will expect tomorrow.

We use an example ("The finite example")



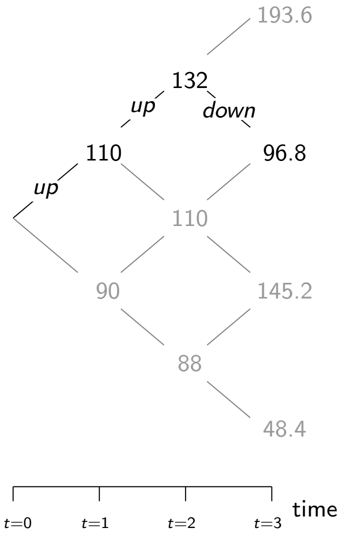
There are three points in time in the future.

The movements **up** and **down** along the path occur with probability 0.5.[a]

Different cash flow realizations along the path might be observed.

[a] "Up" refers to an upward move on the page, not to the absolute level of the cash flow.

A remark: States

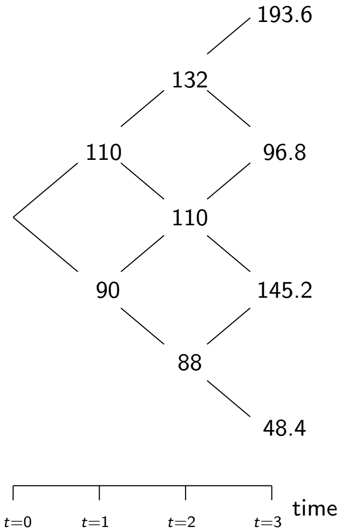


By a “state”, we mean the entire path (entire future). In our example, we describe ω by the sequence of movements, for instance:

up – up – down[a]

(leading to cashflows 110, 132 and 96.8 at $t = 1, 2, 3$).

[a] Formally $\omega = uud$.



Let us consider the cash flow paid at time t=3, i.e., $\widetilde{CF}_3$. What will its **expectation be tomorrow**?

This depends on the state we will have tomorrow. Two cases are possible: $\widetilde{CF}_1 = 110$ or $\widetilde{CF}_1 = 90$.

Case 1 ($\widetilde{CF}_1 = 110$)

$$\Longrightarrow \text{ Expectation of } \widetilde{CF}_3 = \frac{1}{4}\times 193.6 + \frac{2}{4}\times 96.8 + \frac{1}{4}\times 145.2 = 133.1.$$

Case 2 ($\widetilde{CF}_1 = 90$)

$$\Longrightarrow \text{ Expectation of } \widetilde{CF}_3 = \frac{1}{4}\times 96.8 + \frac{2}{4}\times 145.2 + \frac{1}{4}\times 48.4 = 108.9.$$

Hence, expectation of $\widetilde{CF}_3$ is

$$\mathrm{E}_1\left[\widetilde{CF}_3\right] = \begin{cases} 133.1 & \text{if the development at } t = 1 \text{ is up,} \\ 108.9 & \text{if the development at } t = 1 \text{ is down.} \end{cases}$$

The expectation of $\widetilde{CF}_3$ depends on the state of nature at time t=1. Hence, it is conditional: conditional on the information available at time t=1 (abbreviated as $\mathrm{E}_1[\cdot]$).

A conditional expectation captures our view of what we will expect in the future.

This conditional expectation can itself be uncertain.

Arithmetic textbook of

Adam Ries (1492–1559)

How to use conditional expectations? We will not present proofs, but only rules for calculation.

The first three rules will be familiar from classical expectations; two are new.

Rule 1: Classical expectation

$$\boxed{\mathrm{E}_0\left[\widetilde{X}\right] = \mathrm{E}\left[\widetilde{X}\right]}$$

At $t = 0$ conditional expectation and classical expectation coincide.

Or: **conditional expectation generalizes classical expectation.**

Rule 2: Linearity

$$\boxed{\mathrm{E}_t\left[a\widetilde{X}+b\widetilde{Y}\right]=a\,\mathrm{E}_t\left[\widetilde{X}\right]+b\,\mathrm{E}_t\left[\widetilde{Y}\right]}$$

Business as usual . . .

Rule 3: Certain quantity

$\boxed{\mathrm{E}_t[1] = 1}$ Again a generalization of classical expectation.

Using this and linearity we get for risk-free quantities X,

$$
\begin{aligned}
\mathrm{E}_t[X] &= \mathrm{E}_t[X \times 1] \\
&= X \times \mathrm{E}_t[1] && \text{by Rule 2} \\
&= X
\end{aligned}
$$

Rule 4: Iterated expectation

Let $s \geq t$ then

$$\boxed{\mathsf{E}_t\left[\mathsf{E}_s\left[\widetilde{X}\right]\right] = \mathsf{E}_t\left[\widetilde{X}\right]}$$

When we think today about what we will know tomorrow about the day after tomorrow,

we will only know tomorrow what we already believe today that we will know tomorrow.

Rule 5: Known or realized quantities

If $\widetilde{X_t}$ is known at time t

$$\boxed{\mathrm{E}_t\left[\widetilde{X_t}\,\widetilde{Y}\right] = \widetilde{X_t}\,\mathrm{E}_t\left[\widetilde{Y}\right]}$$

We can take out from the expectation what is known.

Or: known quantities are like risk-free quantities.

We want to check our rules by looking at the finite example and an infinite example. We start with the finite example:

Remember that we had

$$\mathsf{E}_1\left[\widetilde{CF}_3\right] = \begin{cases} 133.1 & \text{if up at time } t = 1\ , \\ 108.9 & \text{if down at time } t = 1\ . \end{cases}$$

From this we get

$$\mathsf{E}\left[\mathsf{E}_1\left[\widetilde{CF}_3\right]\right] = \frac{1}{2} \times 133.1 + \frac{1}{2} \times 108.9 = 121.$$

Finite example

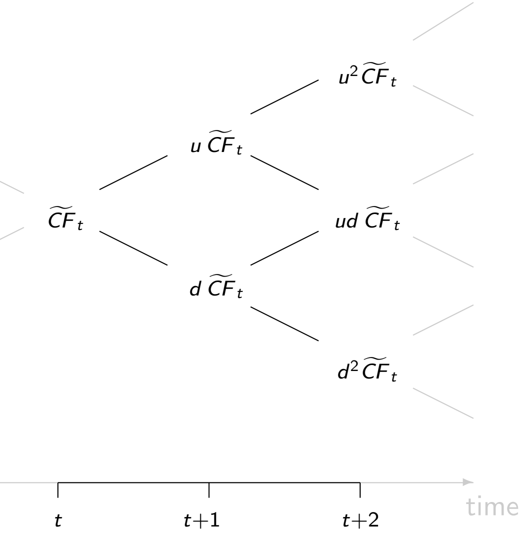
And indeed

$$\begin{aligned} \mathsf{E}\left[\mathsf{E}_1\left[\widetilde{CF}_3\right]\right] &= \mathsf{E}_0\left[\mathsf{E}_1\left[\widetilde{CF}_3\right]\right] && \text{by Rule 1} \\ &= \mathsf{E}_0\left[\widetilde{CF}_3\right] && \text{by Rule 4} \\ &= \mathsf{E}\left[\widetilde{CF}_3\right] && \text{by Rule 1} \\ &= 121\ ! \end{aligned}$$

It seems **purely by chance** that

$$\mathrm{E}_1\left[\widetilde{CF}_3\right] = 1.1^{3-1} \times \widetilde{CF}_1\,,$$

but it is on purpose! This will become clear later (when discussing martingale-like cash flows).



Again two factors up and down with probability p_u and p_d and $0 < d < u$

or

$$\widetilde{CF}_{t+1} = \begin{cases} u\widetilde{CF}_t & \text{up,} \\ d\widetilde{CF}_t & \text{down.} \end{cases}$$

Let us evaluate the conditional expectation

$$\mathrm{E}_t\left[\widetilde{CF}_{t+1}\right] = p_u u \widetilde{CF}_t + p_d d \widetilde{CF}_t = \underbrace{(p_u u + p_d d)}_{:=1+g} \widetilde{CF}_t\,,$$

where g is the expected growth rate.

If $g = 0$ it is said that the cash flows 'form a martingale'. In the infinite example we will later assume this property.

This can be extended if $s > t$

$$
\begin{aligned}
\mathrm{E}_t\left[\widetilde{CF}_s\right] &= \mathrm{E}_t\left[\mathrm{E}_{s-1}\left[\widetilde{CF}_s\right]\right] && \text{by Rule 4} \\
&= \mathrm{E}_t\left[(1+g)\widetilde{CF}_{s-1}\right] && \text{see above} \\
&= (1+g)\,\mathrm{E}_t\left[\widetilde{CF}_{s-1}\right] && \text{by Rule 2} \\
&= (1+g)^{s-t}\,\mathrm{E}_t\left[\widetilde{CF}_t\right] && \text{repeating argument} \\
&= (1+g)^{s-t}\widetilde{CF}_t\,\mathrm{E}_t\left[1\right] && \text{by Rule 5} \\
&= (1+g)^{s-t}\widetilde{CF}_t && \text{by Rule 3}
\end{aligned}
$$

We always stay in the present. Conditional expectation handles our knowledge of the future.

Five rules cover the necessary mathematics.

Two examples will help us understanding our theory.